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On Soft Bitopological Ordered Spaces

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Abstract

This paper introduces soft bitopological ordered spaces, combining soft topological spaces with partial order relations. The authors extensively investigate increasing, decreasing, and balancing pairwise open and closed soft sets, analyzing their properties. They prove that the collection of increasing (decreasing) open soft sets forms an increasing (decreasing) soft topology. The paper thoroughly examines increasing and decreasing pairwise soft closure and interior operators. Notably, it introduces $bi-$ordered soft separation axioms, denoted as PST_i ($PST_i^{\bullet}, PST_i^*$, PST_i^{**})$-$ordered spaces, $i = 0, 1, 2$, showcasing their interrelationships through examples. It explores separation axiom distinctions in bitopological ordered spaces, referencing relevant literature such as the work of El-Shafei et al. [5]. The paper investigates new types of regularity and normality in soft bitopological ordered spaces and their connections to other properties. Importantly, it establishes the equivalence of three properties for a soft bitopological ordered space satisfying the conditions of being TP^*-soft regularly ordered: PST_2-ordered, PST_1-ordered, and PST_0-ordered. It introduces the concept of a $bi-$ordered subspace and explores its hereditary property. The authors define soft bitopological ordered properties using ordered embedding soft homeomorphism maps and verify their applicability for different types of PST_i-ordered spaces, $i = 0, 1, 2$. Finally, the paper identifies the properties of being a $TP^*; (PP^*)-$soft T_3-ordered space and a TP-soft T_4-ordered space as a soft bitopological ordered property.

Keywords: soft bitopological ordered space; increasing (decreasing) pairwise soft closure operator; PST_i (resp. $PST_i^{\bullet}, PST_i^*, PST_i^{**}$)$-$ordered spaces; $(i = 0, 1, 2)$, TP (PP, TP^*, PP^*)$-$soft T_3-ordered spaces; $TP-$soft T_4-ordered space.

1 Introduction

In the field of mathematics, the concept of topological ordered spaces, as introduced by Nachbin [24], is a fundamental framework that combines partial order theory with the principles of topological spaces. Building upon this foundation, researchers such as McCartan [22] explored the application of monotone neighborhoods to investigate ordered separation axioms within topological ordered spaces. Abo-Elhamayel et al. [6] introduced a novel class of separation axioms, leveraging the concept of limit points of a set, thus contributing to the evolving landscape of topological ordered spaces.

In real-life problem-solving, the inherent vagueness and uncertainty have led to the development of mathematical tools like fuzzy sets, intuitionistic fuzzy sets, rough sets, vague sets, and soft sets. Molodtsov's pioneering work [23] introduced the notion of soft sets, offering an effective means to handle these challenges. Subsequently, Maji et al. [21, 20], Aktas and Cagman [1], Senel and Cagman [27, 29], Shabir and Naz [30], and Hussain and Ahmad [14] expanded upon the theory of soft sets, exploring their applications in decision-making and algebraic structures.

Al-shami's work [2] and the research by Tantawy et al. [33] have introduced innovative soft separation axioms, incorporating partial belong and total non-belong relations, and employing the concept of soft points, respectively. Exploring the intricacies of soft neighborhood systems, Zorlutuna et al. [34], Nazmul and Samanta [25], Gocur et al. [13], and Hussain et al. [15] have introduced and examined diverse features associated with soft topological spaces and their separation axioms. Expanding on the understanding of soft topological spaces, Singh and Noorie [32] have further enriched this domain. In 1963, Kelly [18] introduced the concept of a bitopological space, presenting it as a more intricate structure compared to a topological space. More recently, Sharma et al. [31] innovatively introduced a novel form of weak open sets through the process of idealization within the context of bitopological spaces.

El-Sheikh et al. [12] and Ittanagi [16] introduced innovative extensions to soft topological spaces, namely, supra soft topological spaces and soft bitopological spaces. These new concepts are defined over initial universal sets and incorporate fixed sets of parameters, opening up new avenues for exploration. Kandil et al. [17] and Senel et al. [28] further advanced the study of soft bitopological spaces by defining fundamental notions such as pairwise open and closed soft sets, pairwise soft closure, interior, kernel operators, and more. Their work also encompasses the examination of pairwise soft continuous mappings and open and closed soft mappings between two soft bitopological spaces.

The work by El-Shafei et al. [5, 11] has played a pivotal role, making significant contributions through the introduction of monotone soft sets and increasing (decreasing) soft operators. Furthermore, they have established the groundwork for the concept of soft topological ordered spaces and formulated ordered soft separation axioms. El-Shafei et al. [4, 7] brought forth the notion of soft ordered maps and explored the partial belong relation concerning soft separation axioms and decision-making problems. In 2020 [8], they presented two innovative variations of ordered soft separation axioms.

The objective of this study is to establish a soft bitopological ordered space, which integrates a soft bitopological space with a partial order relation. Specifically, in this paper, we treat a generating soft bitopological ordered space and a soft bitopological space as equivalent if the partial order relation corresponds to an equality relation. To facilitate our investigations, we commence by introducing the definitions and results of soft set theory, soft topological spaces, and soft bitopological spaces, as they form the fundamental groundwork for our research.

In Section 3, we introduce the concepts of increasing and decreasing pairwise soft sets, shedding light on their fundamental properties. Additionally, we define and explore the notions of increasing, decreasing, and balancing total and partial pairwise soft neighborhoods, as well as increasing and decreasing pairwise open soft neighborhoods, while illustrating their relationships. Notably, one of the significant findings in Section 3 is Theorem 3.2, which plays a crucial role in verifying results concerning soft topological spaces.

In Section 4, we introduce the concepts of increasing and decreasing pairwise soft closure and interior operators, illustrating their relationships with the help of examples. In Section 5, we present the concepts of $bi-$ordered soft separation axioms, specifically PST_i-ordered spaces, $PST_i^{\bullet}-$ordered spaces, $PST_i^{*}-$ordered spaces, and $PST_i^{**}-$ordered spaces (where $i=0,1,2$). We offer illustrative examples to demonstrate the interrelationships between these axioms. Further exploration of novel patterns of regularity and normality in soft bitopological ordered spaces, along with their interconnections to other characteristics, deepens our comprehension of these spaces. Notably, the paper establishes the equivalence of three properties when $(\Upsilon,\eta_1,\eta_2,\Pi,\lesssim)$ satisfies the conditions of being $TP^{*}-$soft regularly ordered: $(\Upsilon,\eta_1,\eta_2,\Pi,\lesssim)$ is PST_2-ordered, $(\Upsilon,\eta_1,\eta_2,\Pi,\lesssim)$ is PST_1-ordered, and $(\Upsilon,\eta_1,\eta_2,\Pi,\lesssim)$ is PST_0-ordered.

Moreover, we introduce the concept of a $bi-$ordered subspace and explore its hereditary property within the context of soft bitopological ordered spaces. Additionally, we define soft bitopological ordered properties and validate them for PST_i-ordered spaces (where $i=0,1,2$), $PST_i^{\bullet}-$ordered spaces, $PST_i^{*}-$ordered spaces, and $PST_i^{**}-$ordered spaces. We also establish the property of being a $TP^{*}(PP^{*})$soft T_3-ordered space and a $TP-$soft T_4-ordered space as a soft bitopological ordered property. In Section 6, we present the discussion of our paper, and in Section 7, we provide the concluding remarks on our research findings.

2 Preliminaries

This section provides a brief overview of key concepts and relevant results from the fields of soft sets, soft topological spaces, soft bitopological spaces, and soft topological ordered spaces, which will be used in this paper.

From now on, let Υ represent the universe set, Π represent a fixed set of parameters, and 2^{Υ} represent the power set of Υ.

Definition 2.1. [9, 21, 23, 30] *A soft set is defined as a pair (ω,Π), where $\omega:\Pi\to 2^{\Upsilon}$. The notation ω_{Π} is used instead of (ω,Π) for brevity. A soft set can also be represented as a set of ordered pairs, where $\omega_{\Pi}=\{(\alpha,\omega(\alpha)):\alpha\in\Pi,\omega(\alpha)\in 2^{\Upsilon}\}$. The collection of all soft sets over Υ is denoted by $P(\Upsilon)^{\Pi}$. A null soft set, denoted by $\widehat{\phi}$, is one where $\omega(\alpha)=\emptyset$ for all $\alpha\in\Pi$. An absolute soft set, denoted by Υ_{Π}, is one where $\omega(\alpha)=\Upsilon$ for all $\alpha\in\Pi$. Two soft sets, $\omega_{\Pi},\hbar_{\Pi}\in P(\Upsilon)^{\Pi}$, are considered a soft subset, denoted by $\hbar_{\Pi}\sqsubseteq\omega_{\Pi}$, if $\hbar(\alpha)\subseteq\omega(\alpha)$ for all $\alpha\in\Pi$. They are considered equal, denoted by $\hbar_{\Pi}=\omega_{\Pi}$, if $\hbar_{\Pi}\sqsubseteq\omega_{\Pi}$ and $\omega_{\Pi}\sqsubseteq\hbar_{\Pi}$. The union and intersection of two soft sets, $\hbar_{\Pi}$ and ω_{Π}, are represented by $\hbar_{\Pi}\sqcup\omega_{\Pi}$ and $\hbar_{\Pi}\sqcap\omega_{\Pi}$, respectively. The difference of two soft sets, $\hbar_{\Pi}$ and ω_{Π}, is denoted by $\hbar_{\Pi}-\omega_{\Pi}$, and the complement of a soft set $\hbar_{\Pi}$ is denoted by $\hbar_{\Pi}^{c}$.*

Definition 2.2. [25, 26] *A soft set $\hbar_{\Pi}:\Pi\to 2^{\Upsilon}$ defined as $\hbar(e)=\{\rho\}$ if $e=\alpha$ and $\hbar(e)=\emptyset$ if $e\in\Pi-\{\alpha\}$ is called a soft point and denoted by ρ^{α}. The collection of all soft points over Υ is denoted by $Sp(\Upsilon)^{\Pi}$. A soft point ρ^{α} is said to be belonging to a soft set $\hbar_{\Pi}$, denoted by $\rho^{\alpha}\widehat{\in}\hbar_{\Pi}$, if for the member $\alpha\in\Pi$, $\rho(\alpha)\subseteq\hbar(\alpha)$.*

Definition 2.3. [10] *A soft set ω_{Π} over Υ is referred to as a soft singleton if there exists an element ν_0 in*

Υ such that $\omega(\alpha) = \nu_0$ for all α in Π. We denote a soft singleton as $\omega_\Pi^{\nu_0}$.

Definition 2.4. [5, 23] *For a soft set $\hbar_\Pi$ over Υ and an element $\rho \in \Upsilon$, we say $\rho \in \hbar_\Pi$ if $\rho \in \hbar(\alpha)$ for every $\alpha \in \Pi$ and $\rho \notin \hbar_\Pi$ if $\rho \notin \hbar(\alpha)$ for some $\alpha \in \Pi$. We say $\rho \Subset \hbar_\Pi$ if $\rho \in \hbar(\alpha)$ for some $\alpha \in E$ and $a \not\Subset \hbar_\Pi$ if $a \notin \hbar(\alpha)$ for every $\alpha \in \Pi$. The notations $\in, \notin, \Subset$ and $\not\Subset$ are respectively read as belong, non-belong, partial belong and total non-belong relations.*

Definition 2.5. [30] *A soft topology on Υ is a collection of soft sets over Υ under Π that satisfy the following conditions:*

1. *The null soft set and the absolute soft set are included in the collection.*
2. *The union of any collection of soft sets in the collection is also in the collection.*
3. *The intersection of any two soft sets in the collection is also in the collection.*

The triple (Υ, η, Π) is called a soft topological space over Υ, where η is the soft topology. Each member of η is referred to as a soft open set, and its relative complement is called a soft closed set.

Definition 2.6. [34] *A soft subset ε_Π of a soft topological space (Υ, η, Π) is called soft neighborhood of $\nu \in \Upsilon$, if there exists a soft open set ω_Π such that $\nu \in \omega_\Pi \sqsubset \varepsilon_\Pi$.*

Definition 2.7. [25] *Let $P(\Upsilon)^\Pi$ and $P(\Gamma)^K$ be families of soft sets over Υ and Γ, respectively. Let $\phi : \Upsilon \to \Gamma$ and $\psi : \Pi \to K$ be two mappings. The mapping $\phi_\psi : P(\Upsilon)^\Pi \to P(\Gamma)^K$ is a soft mapping from Υ to Γ, denoted by ϕ_ψ, defined as follows:*

1. *For $\omega_\Pi \in P(\Upsilon)^\Pi$, $\phi_\psi(\omega_\Pi)(k) = \bigcup_{\alpha \in \psi^{-1}(k)} \omega(\alpha)$ if $\psi^{-1}(k) \neq \emptyset$, and $\phi_\psi(\omega_\Pi)(k) = \emptyset$ otherwise, for all $k \in K$. The soft set $\phi_\psi(\omega_\Pi)$ is called the soft image of ω_Π.*
2. *For $\lambda_K \in P(\omega)^K$, $\phi_\psi^{-1}(\lambda_K)(\alpha) = \phi^{-1}\big(\lambda(\psi(\alpha))\big)$, for all $\alpha \in \Pi$. The soft set $\phi_\psi^{-1}(\lambda_K)$ is called the soft inverse image of λ_K.*

Definition 2.8. [34] *Let $P(\Upsilon)^\Pi$ and $P(\Gamma)^K$ be two families of soft sets over Υ and Γ, respectively. A soft mapping $\phi_\psi : P(\Upsilon)^\Pi \to P(\Gamma)^K$ is called soft surjective(injective) mapping if ϕ, ψ are surjective (injective) mappings, respectively. A soft mapping which is a soft surjective and soft injective mapping is called a soft bijection mapping.*

Proposition 2.1. [25] *Consider $\phi_\psi : P(\Upsilon)^\Pi \to P(\Gamma)^K$ is a soft map and let ω_Π and λ_K be two soft subsets of $P(\Upsilon)^\Pi$ and $P(\Gamma)^K$, respectively. Then we have the following results:*

1. *$\omega_\Pi \sqsubseteq \phi_\psi^{-1}(\phi_\psi(\omega_\Pi))$ and the equality relation holds if ϕ_ψ is injective.*
2. *$\phi_\psi(\phi_\psi^{-1}(\lambda_K)) \sqsubseteq \lambda_K$ and the equality relation holds if ϕ_ψ is surjective.*

Definition 2.9. [25] *A soft map $\phi_\psi : (\Upsilon, \eta, \Pi) \to (\Gamma, \eta^*, K)$ is said to be:*

1. *Soft continuous if the inverse image of each soft open subset of (Γ, η^*, K) is a soft open subset of (Υ, η, Π).*
2. *Soft open (resp. soft closed) if the image of each soft open (resp. soft closed) subset of (Υ, η, Π) is a soft open (resp. soft closed) subset of (Γ, η^*, K).*
3. *Soft homeomorphism if it is bijective, soft continuous and soft open.*

Definition 2.10. [16, 17] *A quadrable system $(\Upsilon, \eta_1, \eta_2, \Pi)$ is called a soft bitopological space when η_1 and η_2 are soft topologies on the set Υ with a fixed set of parameters Π. A soft set $\hbar_\Pi$ in a soft bitopological space $(\Upsilon, \eta_1, \eta_2, \Pi)$ is called pairwise open soft $(PO-soft)$ if there exists an η_1-open soft set $\hbar^1_\Pi$ and an η_2-open soft set $\hbar^2_\Pi$ such that $\hbar_\Pi = \hbar^1_\Pi \sqcup \hbar^2_\Pi$, and pairwise closed soft $(PC-soft)$ if the complement of $\hbar_\Pi$ is a $PO-$soft set. The family of all $PO-$soft sets, denoted by η_{12}, is a supra soft topological space associated with the soft bitopological space $(\Upsilon, \eta_1, \eta_2, \Pi)$.*

Theorem 2.1. [17] *Let $(\Upsilon, \eta_1, \eta_2, \Pi)$ be a soft bitopological space. Then:*

1. *Each η_j-open soft set is a PO-soft set, $j = 1, 2$, i.e., $\eta_j \subseteq \eta_{12}$.*
2. *Each η_j-closed soft set is a PC-soft set, $j = 1, 2$, i.e., $\eta^c_j \subseteq \eta^c_{12}$.*

Definition 2.11. [19] *A binary relation $\lesssim$ on a set Υ is a partial order relation if it is reflexive, anti-symmetric, and transitive. The equality relation on Υ, denoted by $\blacktriangle$, is defined as $\{(\rho, \rho) : \rho \in \Upsilon\}$.*

Definition 2.12. [24] *A triple $(\Upsilon, \eta, \lesssim)$ is called a topological ordered space when (Υ, η) is a topological space and $(\Upsilon, \lesssim)$ is a partially ordered set.*

Definition 2.13. [5] *A triple $(\Upsilon, \Pi, \lesssim)$ is called a partially ordered soft space when $\lesssim$ is a partial order relation on the set Υ. An increasing soft operator $i : (P(\Upsilon)^\Pi, \lesssim) \to (P(\Upsilon)^\Pi, \lesssim)$ and a decreasing soft operator $d : (P(\Upsilon)^\Pi, \lesssim) \to (P(\Upsilon)^\Pi, \lesssim)$ are defined for each soft set $\hbar_\Pi$ in $P(\Upsilon)^\Pi$ by $i(\hbar_\Pi)(\alpha) = i\hbar(\alpha) = \{\rho \in \Upsilon : \delta \lesssim \rho$, for some $\delta \in \hbar(\alpha)\}$ and $d(\hbar_\Pi)(\alpha) = d\hbar(\alpha) = \{\rho \in \Upsilon : \rho \lesssim \delta$, for some $\delta \in \hbar(\alpha)\}$ respectively. A soft subset $\hbar_\Pi$ of the partially ordered soft space $(\Upsilon, \Pi, \lesssim)$ is called increasing if $\hbar_\Pi = i(\hbar_\Pi)$, decreasing if $\hbar_\Pi = d(\hbar_\Pi)$, and balancing if it is both increasing and decreasing.*

Proposition 2.2. [5] *The following two results hold for a soft map $\phi_\psi : P(\Upsilon)^\Pi \to P(\Gamma)^K$.*

1. *The image of each soft point is soft point.*
2. *If ϕ_ψ is bijective, then the inverse image of each soft point is soft point.*

Definition 2.14. [5] *Let ν^α and ζ^α be two soft points in a partially ordered soft space $(\Upsilon, \Pi, \lesssim)$. Then, $\nu^\alpha \leq \zeta^\alpha$ if $\nu \leq \zeta$.*

Definition 2.15. [5] *A soft map $\phi_\psi : (P(\Upsilon)^\Pi, \lesssim_1) \to (P(\Gamma)^K, \lesssim_2)$ is said to be:*

1. *Increasing if $\nu^\alpha \lesssim_1 \zeta^\alpha$, then $\phi_\psi(\nu^\alpha) \lesssim_2 \phi_\psi(\zeta^\alpha)$.*
2. *Decreasing if $\nu^\alpha \lesssim_1 \zeta^\alpha$, then $\phi_\psi(\zeta^\alpha) \lesssim_2 \phi_\psi(\nu^\alpha)$.*
3. *Ordered embedding if $\nu^\alpha \lesssim_1 \zeta^\alpha$ if and only if $\phi_\psi(\nu^\alpha) \lesssim_2 \phi_\psi(\zeta^\alpha)$.*

Theorem 2.2. [5] *The following two results hold for a soft map $\phi_\psi : \big(P(\Upsilon)^\Pi, \lesssim_1\big) \to \big(P(\Gamma)^K, \lesssim_2\big)$.*

1. *If ϕ_ψ is increasing, then the inverse image of each increasing (resp. decreasing) soft subset of Γ_K is an increasing (resp. a decreasing) soft subset of Υ_Π.*
2. *If ϕ_ψ is decreasing, then the inverse image of each increasing (resp. decreasing) soft subset of Γ_K is an decreasing (resp. a increasing) soft subset of Υ_Π.*

Theorem 2.3. [5] *Let $\phi_\psi : \big(P(\Upsilon)^\Pi, \lesssim_1\big) \to \big(P(\Gamma)^K, \lesssim_2\big)$ be a bijective ordered embedding soft map. Then the image of each increasing (resp. decreasing) soft subset of Υ_Π is an increasing (resp. a decreasing) soft subset of Γ_K.*

Proposition 2.3. [5] *Let $(\Upsilon, \Pi, \lesssim)$ be a partially ordered soft space, and let $\{\hbar_\Pi^\beta : \beta \in \Omega\}$ be a collection of soft sets in $(\Upsilon, \Pi, \lesssim)$. If all the soft sets $\hbar_\Pi^\beta$ are increasing (resp. decreasing), then $\sqcup_{\beta\in\Omega}\hbar_\Pi^\beta$ and $\sqcap_{\beta\in\Omega}\hbar_\Pi^\beta$ are also increasing (resp. decreasing).*

Proposition 2.4. [5] *Let $i : \big(P(\Upsilon)^\Pi, \lesssim\big) \to \big(P(\Upsilon)^\Pi, \lesssim\big)$ and $d : \big(P(\Upsilon)^\Pi, \lesssim\big) \to \big(P(\Upsilon)^\Pi, \lesssim\big)$ be increasing and decreasing soft operators, and let $\hbar_\Pi$ and ω_Π be two soft sets in $(\Upsilon, \Pi, \lesssim)$. Then:*

1. $i(\widehat{\phi}) = \widehat{\phi}$ *and* $d(\widehat{\phi}) = \widehat{\phi}$.
2. $\hbar_\Pi \sqsubseteq i(\hbar_\Pi)$ *and* $\hbar_\Pi \sqsubseteq d(\hbar_\Pi)$.
3. $i(i(\hbar_\Pi)) = i(\hbar_\Pi)$ *and* $d(d(\hbar_\Pi)) = d(\hbar_\Pi)$
4. $i[\hbar_\Pi \sqcup \omega_\Pi] = i(\hbar_\Pi) \sqcup i(\omega_\Pi)$*and* $d[\hbar_\Pi \sqcup \omega_\Pi] = d(\hbar_\Pi) \sqcup d(\omega_\Pi)$.

Definition 2.16. [5] *A quadrable system $(\Upsilon, \eta, \Pi, \lesssim)$ can be rephrased as a soft topological ordered space ($STOS$) if (Υ, η, Π) is a soft topological space and $(\Upsilon, \Pi, \lesssim)$ is a partially ordered soft space. A soft set $\hbar_\Pi$ in a soft topological ordered space $(\Upsilon, \eta, \Pi, \lesssim)$ is called increasing (decreasing) open soft if it is soft open and increasing (decreasing).*

Definition 2.17. [5] *A soft subset ε_Π of an $STOS$ $(\Upsilon, \eta, \Pi, \lesssim)$ is called an increasing (resp. a decreasing) soft neighborhood of $\nu \in \Upsilon$ if ε_Π is soft neighborhood of ν and increasing (resp. decreasing).*

Definition 2.18. [3] *A quadrable system $(\Upsilon, \eta, \Pi, \lesssim)$ is referred to as a supra soft topological ordered space, if (Υ, η, Π) is a supra soft topological space and $(\Upsilon, \Pi, \lesssim)$ is a partially ordered soft space.*

Definition 2.19. [5] *Let $(\Upsilon, \eta, \Pi, \lesssim)$ be an $STOS$. We say it satisfies the following properties:*

1. *It is lower (resp. upper) P-soft T_1-ordered if for any distinct points $\nu, \zeta \in \Upsilon$, there exists an increasing (resp. decreasing) soft neighborhood ε_Π of ν such that $\zeta \notin \varepsilon_\Pi$.*
2. *It is P-soft T_0-ordered if it is either lower P-soft T_1-ordered or upper P-soft T_1-ordered.*
3. *It is P-soft T_1-ordered if it is both lower P-soft T_1-ordered and upper P-soft T_1-ordered.*
4. *It is P-soft T_2-ordered if for any distinct points $\nu, \zeta \in \Upsilon$, there exist disjoint soft neighborhoods ε_Π and V_Π of ν and ζ respectively, such that ε_Π is increasing and V_Π is decreasing.*

3 Soft Bitopological Ordered Spaces

This section examines the concepts of soft bitopological ordered spaces, pairwise open and closed soft sets that are increasing, decreasing, or balanced, and pairwise soft neighborhoods that are increasing, decreasing, total, or partial, as well as the properties of these concepts in a soft bitopological ordered space.

Proposition 3.1. *If ω_Π and $\hbar_\Pi$ are two soft sets in $P(\Upsilon)^\Pi$ and i and d are increasing and decreasing soft operators respectively, then:*

1. $i(\Upsilon_\Pi) = \Upsilon_\Pi$ *and* $d(\Upsilon_\Pi) = \Upsilon_\Pi$
2. *If* $\omega_\Pi \sqsubseteq \hbar_\Pi$, *then* $i(\omega_\Pi) \sqsubseteq i(\hbar_\Pi)$ *and* $d(\omega_\Pi) \sqsubseteq d(\hbar_\Pi)$
3. $i[\omega_\Pi \sqcap \hbar_\Pi] \sqsubseteq i(\omega_\Pi) \sqcap i(\hbar_\Pi)$

4. $d[\omega_\Pi \sqcap \hbar_\Pi] \sqsubseteq d(\omega_\Pi) \sqcap d(\hbar_\Pi)$

Proof. The proof for the first and third cases is given, and the proof for the second and fourth cases can be done similarly.

1. $i(\Upsilon_\Pi)(\alpha) = i(\Upsilon(\alpha)) = i(\Upsilon) = \Upsilon = \Upsilon(\alpha)$. Therefore, $i(\Upsilon_\Pi) = \Upsilon_\Pi$. Similarly, $d(\Upsilon_\Pi) = \Upsilon_\Pi$.

3. $[\omega_\Pi \sqcap \hbar_\Pi] \sqsubseteq \omega_\Pi \sqsubseteq i(\omega_\Pi)$ and $[\omega_\Pi \sqcap \hbar_\Pi] \sqsubseteq \hbar_\Pi \sqsubseteq i(\hbar_\Pi)$, by Proposition 2.4.
Thus, $i[\omega_\Pi \sqcap \hbar_\Pi] \sqsubseteq i(i(\omega_\Pi)) = i(\omega_\Pi)$ and $i[\omega_\Pi \sqcap \hbar_\Pi] \sqsubseteq i(i(\hbar_\Pi)) = i(\hbar_\Pi)$.
Thus, $i[\omega_\Pi \sqcap \hbar_\Pi] \sqsubseteq i(\omega_\Pi) \sqcap i(\hbar_\Pi)$.

□

The following example illustrates that the equality stated in items 3 and 4 of Proposition 3.1 is not always true.

Example 3.1. *Let $\Pi = \{\alpha_1, \alpha_2\}$, $\lesssim = \blacktriangle \cup \{(\delta, f), (\sigma, \varsigma)\}$ be a partial order relation on $\Upsilon = \{\rho, \delta, \sigma, \varsigma, f\}$ and $\omega_\Pi, \hbar_\Pi$ be two soft sets in $P(\Upsilon)^\Pi$ and defined as follows:*

$$\omega_\Pi = \big\{(\alpha_1, \{\rho, \delta\}), (\alpha_2, \{\rho, \varsigma\})\big\}, \qquad \hbar_\Pi = \big\{(\alpha_1, \{\delta, \sigma\}), (\alpha_2, \{\rho, \delta, \sigma\})\big\}.$$

Then,

$$i(\omega_\Pi) = \big\{(\alpha_1, \{\rho, \delta, f\}), (\alpha_2, \{\rho, \varsigma\})\big\} \text{ and } i(\hbar_\Pi) = \big\{(\alpha_1, \{\delta, \sigma, \varsigma, f\}), (\alpha_2, \Upsilon)\big\}.$$

Therefore,

$$i(\omega_\Pi \sqcap \hbar_\Pi) = \big\{(\alpha_1, \{\delta, f\}), (\alpha_2, \{\rho\})\big\} \sqsubseteq i(\omega_\Pi) \sqcap i(\hbar_\Pi) = \big\{(\alpha_1, \{\delta, f\}), (\alpha_2, \{\rho, \varsigma\})\big\}.$$

Also,

$$d(\omega_\Pi) = \big\{(\alpha_1, \{\rho, \delta\}), (\alpha_2, \{\rho, \sigma, \varsigma\})\big\} \text{ and } d(\hbar_\Pi) = \big\{(\alpha_1, \{\delta, \sigma\}), (\alpha_2, \{\rho, \delta, \sigma\})\big\}.$$

Therefore,

$$d(\omega_\Pi \sqcap \hbar_\Pi) = \big\{(\alpha_1, \{\delta\}), (\alpha_2, \{\rho\})\big\} \sqsubseteq d(\omega_\Pi) \sqcap d(\hbar_\Pi) = \big\{(\alpha_1, \{\delta\}), (\alpha_2, \{\rho, \sigma\})\big\}.$$

Definition 3.1. *The system composed of a set Υ, two topologies η_1 and η_2, a set Π, and a partial order relation $\lesssim$ is called a soft bitopological ordered space ($SBTOS$) if it satisfies two conditions:*

1. *$(\Upsilon, \eta_1, \eta_2, \Pi)$ is a soft bitopological space.*
2. *$(\Upsilon, \Pi, \lesssim)$ is a partially ordered soft space.*

Definition 3.2. *In an $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, a soft set ω_Π over Υ can be classified into different types. These types include:*

1. *Increasing pairwise open soft ($IPO-soft$) if $\omega_\Pi = \omega^1_\Pi \sqcup \omega^2_\Pi$, $\omega^\beta_\Pi \in \eta_\beta$ and increasing, $\beta = 1, 2$.*
2. *Decreasing pairwise open soft ($DPO-soft$) if $\omega_\Pi = \omega^1_\Pi \sqcup \omega^2_\Pi$, $\omega^\beta_\Pi \in \eta_\beta$ and decreasing, $\beta = 1, 2$.*
3. *Increasing pairwise closed soft ($IPC-soft$) if $\omega_\Pi = \omega^1_\Pi \sqcap \omega^2_\Pi$, $\omega^\beta_\Pi \in \eta^c_\beta$ and increasing, $\beta = 1, 2$.*

4. *Decreasing pairwise closed soft* $(DPO-soft)$ *if* $\omega_\Pi = \omega^1_\Pi \sqcap \omega^2_\Pi$, $\omega^\beta_\Pi \in \eta^c_\beta$ *and decreasing,* $\beta = 1, 2$.
5. *Balancing pairwise open soft* $(BPO-soft)$*: a soft set that is both* $IPO-soft$ *and* $DPO-soft$.
6. *Balancing pairwise closed soft* $(BPC-soft)$*: a soft set that is both* $IPC-soft$ *and* $DPC-soft$.

The collection of all $IPO-soft$ *sets in* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is denoted by* $IPOS(\Upsilon, \eta_1, \eta_2)_\Pi$*, and similarly for* $DPO-soft$ *sets,* $IPC-soft$ *sets and* $DPC-soft$ *sets.*

Proposition 3.2. *For a* $PO(PC)-soft$ *set* ω_Π *in an* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *and an increasing soft operator* i*, the following holds:*

1. *The original soft set is a soft subset of the result of applying the increasing soft operator,* $\omega_\Pi \sqsubseteq i(\omega_\Pi)$.
2. *Applying the increasing soft operator twice results in the same soft set as applying it once,* $i(i(\omega_\Pi)) = i(\omega_\Pi)$.

Proof. We will only provide proof for certain cases of the two statements mentioned above, and that the remaining cases, which are enclosed in parentheses, can be proved in a similar manner.

1.

$$\begin{aligned}
\omega_\Pi &= \omega^1_\Pi \sqcup \omega^2_\Pi, \quad \omega^\beta_\Pi \text{ are } \eta_\beta-\text{increasing}, \quad \beta = 1, 2,\\
&\sqsubseteq i(\omega^1_\Pi) \sqcup i(\omega^2_\Pi), \quad \omega^\beta_\Pi \sqsubseteq i(\omega^\beta_\Pi), \quad \beta = 1, 2,\\
&= i\big[\omega^1_\Pi \sqcup \omega^2_\Pi\big], \quad \text{by Proposition 2.4},\\
&= i(\omega_\Pi).
\end{aligned}$$

2.

$$\begin{aligned}
i(i(\omega_\Pi)) &= i\Big[i\big[\omega^1_\Pi \sqcup \omega^2_\Pi\big]\Big], \quad \omega^\beta_\Pi \text{ are } \eta_\beta-\text{increasing}, \quad \beta = 1, 2,\\
&= i\Big[i(\omega^1_\Pi) \sqcup i(\omega^2_\Pi)\Big], \quad \text{by Proposition 2.4},\\
&= i\big[i(\omega^1_\Pi)\big] \sqcup i\big[i(\omega^2_\Pi)\big],\\
&= i(\omega^1_\Pi) \sqcup i(\omega^2_\Pi),\\
&= i\Big[\omega^1_\Pi \sqcup \omega^2_\Pi\Big],\\
&= i(\omega_\Pi).
\end{aligned}$$

□

A similar proof can be applied to the following proposition.

Proposition 3.3. *For a* $PO(PC)-soft$ *set* ω_Π *in an* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *and a decreasing soft operator* d*, it can be shown that* $\omega_\Pi \sqsubseteq d(\omega_\Pi)$ *and* $d(d(\omega_\Pi)) = d(\omega_\Pi)$.

Theorem 3.1. *In an* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$*, a soft set* ω_Π *is* $IPO(DPO)-soft$ *if and only if* ω^c_Π *is* $DPC(IPC)-soft$.

Proof.
Necessity: Let, ω_Π be an $IPO-$soft set. Then, $\omega_\Pi = \omega^1_\Pi \sqcup \omega^2_\Pi, \omega^\beta_\Pi \in \eta_\beta$ and increasing, $\beta = 1, 2$. This implies that, $\omega^c_\Pi = \left[\omega^1_\Pi \sqcup \omega^2_\Pi\right]^c = \omega^{1c}_\Pi \sqcap \omega^{2c}_\Pi, \omega^{\beta c}_\Pi \in \eta^c_\beta$ and decreasing, $\beta = 1, 2$. Now, $d(\omega^c_\Pi) = d\left[\omega^{1c}_\Pi \sqcap \omega^{2c}_\Pi\right] \sqsubseteq d(\omega^{1c}_\Pi) \sqcap d(\omega^{2c}_\Pi) = \omega^c_\Pi$, by Proposition 3.1. But, $\omega^c_\Pi \sqsubseteq d(\omega^c_\Pi)$, by Proposition 3.3. Therefore, $\omega^c_\Pi = d(\omega^c_\Pi)$. Hence, ω^c_Π is a $DPC-$soft set.

Sufficiency: If ω_Π is a $DPC-$soft set, then $\omega_\Pi = \omega^1_\Pi \sqcap \omega^2_\Pi, \omega^\beta_\Pi \in \eta^c_\beta$ and decreasing, $\beta = 1, 2$. Thus, $\omega^c_\Pi = [\omega^1_\Pi \sqcap \omega^2_\Pi]^c = \omega^{1c}_\Pi \sqcup \omega^{2c}_\Pi, \omega^{\beta c}_\Pi \in \eta^c_\beta$ and increasing, $\beta = 1, 2$. Therefore, ω^c_Π is an $IPO-$soft set.

The proof demonstrates that if ω_Π is $IPO-$soft, then ω^c_Π is $DPC-$soft and vice versa. The same applies for the case between parentheses. □

Definition 3.3. *In a soft topological ordered space* $(\Upsilon, \eta, \Pi, \lesssim)$*, it is called an increasing (a decreasing) soft topological space if all soft open sets in it are increasing (decreasing).*

Theorem 3.2. *In an* $STOS$ $(\Upsilon, \eta, \Pi, \lesssim)$*, the collection of all increasing open soft and decreasing open soft sets forms the increasing soft topology, denoted by* η^I*, and decreasing soft topology, denoted by* η^D*, respectively on* Υ*. i. e.,*

1. $\eta^I = \{\omega_\Pi : \omega_\Pi \in \eta, \omega_\Pi \text{ is increasing}\}$,
2. $\eta^D = \{\omega_\Pi : \omega_\Pi \in \eta, \omega_\Pi \text{ is decreasing}\}$.

Proof.

1. - $\widehat{\phi}, \Upsilon_\Pi$ are increasing open soft sets (clear).Then, $\widehat{\phi}, \Upsilon_\Pi \in \eta^I$.
   - Let $\omega^1_\Pi, \omega^2_\Pi \in \eta^I$. Then, ω^1_Π and ω^2_Π are increasing open soft sets. So, $\omega^1_\Pi \sqcap \omega^2_\Pi$ is increasing open soft, by Proposition 2.3 and Definition 2.5. Therefore, $\omega^1_\Pi \sqcap \omega^2_\Pi \in \eta^I$.
   - Let $\{\omega^\beta_\Pi, \beta \in \Omega\} \subseteq \eta^I$. Then, $\sqcup_{\beta\in\Omega}\omega^\beta_\Pi$ is increasing open soft set, by Proposition 2.3 and Definition 2.5. Therefore, $\sqcup_{\beta\in\Omega}\omega^\beta_\Pi \in \eta^I$.

   Hence, η^I is an increasing soft topology over Υ.

   By analogy with (1), one can prove (2).

□

Corollary 3.1. *For an* $STOS$ $(\Upsilon, \eta, \Pi, \lesssim)$*, we have:*

1. $\eta^{cI} = \left\{\omega^c_\Pi : \omega_\Pi \in \eta^D\right\}$.
2. $\eta^{cD} = \left\{\hbar^c_\Pi : \hbar_\Pi \in \eta^I\right\}$.

Lemma 3.1. *For an* $STOS$ $(\Upsilon, \eta, \Pi, \lesssim)$*, we have:*

1. $\eta^{Ic} = \eta^{cD}$.
2. $\eta^{Dc} = \eta^{cI}$.

Proof.

1. $\omega_\Pi \in \eta^{Ic} \Leftrightarrow \omega_\Pi^c \in \eta^I \Leftrightarrow \omega_\Pi \in \eta^{cD}$.
   By analogy with (1), one can prove (2).

□

Definition 3.4. *A quadrable system $(\Upsilon, \eta, \Pi, \lesssim)$ is defined as an increasing (decreasing) supra soft topological ordered space if it satisfies two conditions: 1) it is a supra soft topological space and 2) every open soft set in $(\Upsilon, \eta, \Pi, \lesssim)$ is an increasing (decreasing) open soft set.*

Corollary 3.2. *For an $STOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, the family of all $IPO-$soft and $DPO-$soft sets forms an increasing supra soft topology, denoted by η_{12}^{IP}, and decreasing supra soft topology, denoted by η_{12}^{DP}, respectively on Υ. i.e.,*

$$\eta_{12}^{IP} = \Big\{\omega_\Pi : \omega_\Pi = \omega_\Pi^1 \sqcup \omega_\Pi^2, \quad \omega_\Pi^\beta \in \eta_\beta \text{ and increasing}, \quad \beta = 1, 2\Big\},$$
$$\eta_{12}^{DP} = \Big\{\hbar_\Pi : \hbar_\Pi = \hbar_\Pi^1 \sqcup \hbar_\Pi^2, \quad \hbar_\Pi^\beta \in \eta_\beta \text{ and decreasing}, \quad \beta = 1, 2\Big\}.$$

However,

$$\eta_{12}^{cIP} = \Big\{\lambda_\Pi^c : \lambda_\Pi \in \eta_{12}^{DP}\Big\},$$
$$\eta_{12}^{cDP} = \Big\{O_\Pi^c : O_\Pi \in \eta_{12}^{IP}\Big\}.$$

Lemma 3.2. *Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $STOS$. Then:*

1. $\eta_{12}^{IPc} = \eta_{12}^{cDP}$.
2. $\eta_{12}^{DPc} = \eta_{12}^{cIP}$.

Proof.

1. $\omega_\Pi \in \eta_{12}^{IPc} \Leftrightarrow \omega_\Pi^c \in \eta_{12}^{IP} \Leftrightarrow \omega_\Pi \in \eta_{12}^{cDP}$.
   Using analogy with statement (1), we can establish the equivalence of statement (2).

□

Theorem 3.3. *Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $STOS$. Then:*

1. *$\widehat{\phi}$ and Υ_Π are $IPO(DPO)-$soft sets and $IPC(DPC)-$soft sets.*
2. *An arbitrary union of $IPO(DPO)-$soft sets is an $IPO(DPO)-$soft set.*
3. *An arbitrary intersection of $IPC(DPC)-$soft sets is an $IPC(DPC)-$soft set.*

Proof.

1. Clear.
2. Let $\{\omega_\Pi^\beta : \beta \in \Omega\} \subseteq IPOS(\Upsilon, \eta_1, \eta_2)_\Pi$. Then, ω_Π^β is an $IPO-$soft set $\forall \beta \in \Omega$, implies there exist two increasing soft sets $\omega_\Pi^{1\beta} \in \eta_1$ and $\omega_\Pi^{2\beta} \in \eta_2$ such that $\omega_\Pi^\beta = \omega_\Pi^{1\beta} \sqcup \omega_\Pi^{2\beta}$, $\forall \beta \in \Omega$ which implies that $\sqcup_{\beta\in\Omega}(\omega_\Pi^\beta) = \sqcup_{\beta\in\Omega}[\omega_\Pi^{1\beta} \sqcup \omega_\Pi^{2\beta}] = [\sqcup_{\beta\in\Omega}\omega_\Pi^{1\beta}] \sqcup [\sqcup_{\beta\in\Omega}\omega_\Pi^{2\beta}]$. Now, since η_1 and η_2 are two soft topologies, then $[\sqcup_{\beta\in\Omega}(\omega_\Pi^{1\beta})] \in \eta_1$ and $[\sqcup_{\beta\in\Omega}(\omega_\Pi^{2\beta})] \in \eta_2$, by Proposition 2.3. Consequently, $\sqcup_{\beta\in\Omega}\omega_\Pi^\beta$ is an $IPO-$soft set.
   Similarly, if $\{\omega_\Pi^\beta : \beta \in \Omega\} \subseteq DPOS(\Upsilon, \eta_1, \eta_2)_\Pi$, then $\sqcup_{\beta\in\Omega}\omega_\Pi^\beta$ is a $DPO-$soft set.
3. Let $\{\omega_\Pi^\beta : \beta \in \Omega\} \subseteq IPCS(\Upsilon, \eta_1, \eta_2)_\Pi$. Then, ω_Π^β is an $IPC-$soft set $\forall \beta \in \Omega$, implies there exist two increasing soft sets $\omega_\Pi^{1\beta} \in \eta_1^c$, $\omega_\Pi^{2\beta} \in \eta_2^c$ such that $\omega_\Pi^\beta = \omega_\Pi^{1\beta} \sqcap \omega_\Pi^{2\beta}$, $\forall \beta \in \Omega$ which implies that $\sqcap_{\beta\in\Omega}(\omega_\Pi^\beta) = \sqcap_{\beta\in\Omega}[\omega_\Pi^{1\beta} \sqcap \omega_\Pi^{2\beta}] = [\sqcap_{\beta\in\Omega}\omega_\Pi^{1\beta}] \sqcap [\sqcap_{\beta\in\Omega}\omega_\Pi^{2\beta}]$. Now, since of η_1 and η_2 are two soft topologies, then $[\sqcap_{\beta\in\Omega}(\omega_\Pi^{1\beta})] \in \eta_1^c$ and $[\sqcap_{\beta\in\Omega}(\omega_\Pi^{2\beta})] \in \eta_2^c$. Consequently, $\sqcap_{\beta\in\Omega}\omega_\Pi^\beta$ is an $IPC-$soft set.
   Similarly, if $\{\omega_\Pi^\beta : \beta \in \Omega\} \subseteq DPCS(\Upsilon, \eta_1, \eta_2)_\Pi$. Then, $\sqcap_{\beta\in\Omega}\omega_\Pi^\beta$ is a $DPC-$soft set.

□

The following example illustrates that:

1. $\eta_{12}^{IP}(\eta_{12}^{DP})$ is not necessarily an increasing (decreasing) soft topology.
2. The intersection of a finite number of $IPO(DPO)-$soft sets may not be an $IPO(DPO)-$soft set.
3. The union of an arbitrary number of $IPC(DPC)-$soft sets may not be an $IPC(DPC)-$soft set.

Example 3.2. *Let $\Pi = \{\alpha_1, \alpha_2\}$ and $\lesssim = \blacktriangle \cup \{(1, \nu) : \nu \in \{2, 3\}\}$ be a partial order relation on the set of natural numbers $\aleph$ and $\eta_1 = \{\omega_\Pi^n : n = 1, 2, 3, ...\} \cup \{\widehat{\phi}, \aleph_\Pi\}$, $\eta_2 = \{\hbar_\Pi^m : m = 1, 2, 3, ...\} \cup \{\widehat{\phi}, \aleph_\Pi\}$ where, ω_Π^n is a soft set over $\aleph$ defined as $\omega^n : \Pi \to 2^\aleph$ such that, $\omega^n(\alpha_1) = \{n, n+1, n+2, ...\}$, $\omega^n(\alpha_2) = \emptyset, \forall n \in \aleph$ and $\hbar_\Pi^m$ is a soft set over $\aleph$ defined as, $\hbar^m : \Pi \to 2^\aleph$ such that, $\hbar^m(\alpha_1) = \{1, 2, 3, ..., m\}$, $\hbar^m(\alpha_2) = \emptyset, \forall n \in \aleph$. Then η_1, η_2 are soft topologies on $\aleph$. Consequently $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ is a soft bitopological ordered space.*

On the one hand, $\omega_\Pi^3 \in \eta_1$, $\hbar_\Pi^3 \in \eta_2$ are $IPO-$soft sets. But $F(\alpha_1) = \omega^3(\alpha_1) \cap \hbar^3(\alpha_1) = \{3, 4, 5, ...\} \cap \{1, 2, 3\} = \{3\}$, $F(\alpha_2) = \omega^3(\alpha_2) \cap \hbar^3(\alpha_2) = \emptyset$. It is clear that F_Π can not be expressed as a union of two increasing soft sets one belongs to η_1 and the other belongs to η_2, i. e., F_Π is not $IPO-$soft set. Consequently, η_{12} is not an increasing soft topology in general.

On the other hand, since ω_Π^3 and $\hbar_\Pi^3$ are $IPO-$soft sets, then ω_Π^{3c} and $\hbar_\Pi^{3c}$ are $DPC-$soft sets, but $\omega_\Pi^{3c} \sqcup \hbar_\Pi^{3c}$ is not $DPC-$soft set, because $\omega_\Pi^{3c} \sqcup \hbar_\Pi^{3c} = G_\Pi$ such that $G(\alpha_1) = \omega^{3c}(\alpha_1) \cup \hbar^{3c}(\alpha_1) = (\aleph - \{3, 4, 5, ...\}) \cup (\aleph - \{1, 2, 3\}) = \aleph - \{3\}$, $G(\alpha_2) = \omega^{3c}(\alpha_2) \cup \hbar^{3c}(\alpha_2) = \aleph$. It is clear that G_Π can not be expressed as an intersection of two decreasing soft sets one belongs to η_1^c and the other belongs to η_2^c, i. e., G_Π is not $DPC-$soft set. Therefore the arbitrary union of $DPC-$soft sets need not be a $DPC-$soft set.

Theorem 3.4. *Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $STOS$. Then:*

1. *$\eta_1^I \cup \eta_1^D \subseteq \eta_1$.*

2. $\eta_2^I \cup \eta_2^D \subseteq \eta_2$.
3. $\eta_1^I \cup \eta_2^I \subseteq \eta_{12}^{IP}$.
4. $\eta_1^D \cup \eta_2^D \subseteq \eta_{12}^{DP}$.
5. $\eta_{12}^{IP} \cup \eta_{12}^{DP} \subseteq \eta_{12}$.

Proof. It is clear that:

1. $\eta_1^I \subseteq \eta_1$ and $\eta_1^D \subseteq \eta_1$ which implies $\eta_1^I \cup \eta_1^D \subseteq \eta_1$.
2. $\eta_2^I \subseteq \eta_2$ and $\eta_2^D \subseteq \eta_2$ which implies $\eta_2^I \cup \eta_2^D \subseteq \eta_2$.
3. $\eta_1^I \subseteq \eta_{12}^{IP}$ and $\eta_2^I \subseteq \eta_{12}^{IP}$ which implies $\eta_1^I \cup \eta_2^I \subseteq \eta_{12}^{IP}$.
4. $\eta_1^D \subseteq \eta_{12}^{DP}$ and $\eta_2^D \subseteq \eta_{12}^{DP}$ which implies $\eta_1^D \cup \eta_2^D \subseteq \eta_{12}^{DP}$.
5. $\eta_{12}^{IP} \subseteq \eta_{12}$ and $\eta_{12}^{DP} \subseteq \eta_{12}$ which implies $\eta_{12}^{IP} \cup \eta_{12}^{DP} \subseteq \eta_{12}$.

□

Definition 3.5. *A soft set ω_Π in an $STOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is called:*

1. *Increasing pairwise open soft set totally containing $\rho \in \Upsilon$ if ω_Π is increasing, $PO-$soft set and $\rho \in \omega_\Pi$.*
2. *Increasing pairwise open soft set partially containing $\rho \in \Upsilon$ if ω_Π is increasing, $PO-$soft set and $\rho \Subset \omega_\Pi$.*
3. *Decreasing pairwise open soft set totally containing $\rho \in \Upsilon$ if ω_Π is decreasing, $PO-$soft set and $\rho \in \omega_\Pi$.*
4. *Decreasing pairwise open soft set partially containing $\rho \in \Upsilon$ if ω_Π is decreasing, $PO-$soft set and $\rho \Subset \omega_\Pi$.*

Definition 3.6. *A soft set ε_Π in an $STOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is said to be:*

1. *A total pairwise soft neighborhood of an element $\rho \in \Upsilon$, if there is a $PO-$soft set ω_Π such that $\rho \in \omega_\Pi \sqsubseteq \varepsilon_\Pi$.*
2. *A partial pairwise soft neighborhood of an element $\rho \in \Upsilon$ if there is a $PO-$soft set ω_Π such that $\rho \Subset \omega_\Pi \sqsubseteq \varepsilon_\Pi$.*

Definition 3.7. *A soft set ε_Π in an $STOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is said to be:*

1. *An increasing total pairwise soft neighborhood ($ITPS-$nbd) of an element $\rho \in \Upsilon$, if ε_Π is a total pairwise soft neighborhood of ρ and is increasing.*
2. *An increasing partial pairwise soft neighborhood ($IPPS-$nbd) of an element $\rho \in \Upsilon$, if ε_Π is a partial pairwise soft neighborhood of ρ and is increasing.*
3. *A decreasing total pairwise soft neighborhood ($DTPS-$nbd) of an element $\rho \in \Upsilon$, if ε_Π is a total pairwise soft neighborhood of ρ and is decreasing.*

4. *A decreasing partial pairwise soft neighborhood* ($DPPS-nbd$) *of an element* $\rho \in \Upsilon$*, if* ε_Π *is a partial pairwise soft neighborhood of* ρ *and is decreasing.*

5. *A balancing total pairwise soft neighborhood* ($BTPS-nbd$) *of an element* $\rho \in \Upsilon$*, if* ε_Π *is a total pairwise soft neighborhood of* ρ *and is balancing.*

6. *A balancing partial pairwise soft neighborhood* ($BPPS-nbd$) *of an element* $\rho \in \Upsilon$*, if* ε_Π *is a partial pairwise soft neighborhood of* ρ *and is balancing.*

Definition 3.8. *Let* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *be an* $STOS$ *and let* $\omega_\Pi \in P(\Upsilon)^\Pi$ *and* $\rho^\alpha \in Sp(\Upsilon)^\Pi$*. Then,* ω_Π *is called:*

1. *An increasing pairwise open soft neighborhood* ($IPS - nbd$) *of* ρ^α*, if there exists a* $PO-$*soft set* λ_Π *such that* $\rho^\alpha \widehat{\in} \lambda_\Pi \sqsubseteq \omega_\Pi$ *and* ω_Π *is increasing.*

2. *A decreasing pairwise open soft neighborhood* ($DPS - nbd$) *of* ρ^α*, if there exists a* $PO-$*soft set* λ_Π *such that* $\rho^\alpha \widehat{\in} \lambda_\Pi \sqsubseteq \omega_\Pi$ *and* ω_Π *is decreasing.*

The following example illustrates the distinction between pairwise open soft sets and pairwise soft neighborhoods, specifically in terms of increasing and decreasing.

Example 3.3. *In this example, let* $\Pi = \{\alpha_1, \alpha_2\}$ *be a set and let* $\lesssim = \blacktriangle \cup \{(\delta, f), (\sigma, \varsigma)\}$ *be a partial order relation on the set* $\Upsilon = \{\rho, \delta, \sigma, \varsigma, f\}$*. Also, let* $\eta_1 = \{\Upsilon_\Pi, \widehat{\phi}, \omega_\Pi\}$ *where* $\omega_\Pi = \{(\alpha_1, \{\rho, \delta\}), (\alpha_2, \{\rho, \varsigma\})\}$ *and* $\eta_2 = \{\Upsilon_\Pi, \widehat{\phi}, \hbar_\Pi\}$ *where* $\hbar_\Pi = \{(\alpha_1, \{\sigma\}), (\alpha_2, \{\varsigma\})\}$*.*

Then, $\eta_{12} = \{\Upsilon_\Pi, \widehat{\phi}, \omega_\Pi, \hbar_\Pi, \lambda_\Pi\}$ *where* $\lambda_\Pi = \{(\alpha_1, \{\rho, \delta, \sigma\}), (\alpha_2, \{\rho, \varsigma\})\}$*. Now,* $i(\omega_\Pi) = \{(\alpha_1, \{\rho, \delta, f\}), (\alpha_2, \{\rho, \varsigma\})\} \neq \omega_\Pi$ *and* $d(\omega_\Pi) = \{(\alpha_1, \{\rho, \delta\}), (\alpha_2, \{\rho, \sigma, \varsigma\})\} \neq \omega_\Pi$*. So* ω_Π *is neither increasing nor decreasing. On the other hand,* $O_\Pi = \{(\alpha_1, \{\rho, \delta, f\}), (\alpha_2, \{\rho, \sigma, \varsigma\})\}$ *is a* $BTPS-$*nbd of* ρ *because:*

1. $\rho \in \omega_\Pi \sqsubseteq O_\Pi$,

2. $i(O_\Pi) = O_\Pi = d(O_\Pi)$.

Furthermore, $K_\Pi = \{(\alpha_1, \{\rho, \delta, f\}), (\alpha_2, \{\rho, \varsigma\})\}$ *is an* $ITPS-$*nbd of* ρ*, but it is not a* $DTPS-$*nbd of* ρ*; and* $V_\Pi = \{(\alpha_1, \{\rho, \delta\}), (\alpha_2, \{\rho, \sigma, \varsigma\})\}$ *is a* $DTPS-$*nbd of* ρ*, but it is not an* $ITPS-$*nbd of* ρ*.*

In this example, K_Π is an $IPS - nbd$ of ρ^{α_1}, for $\rho^{\alpha_1} \widehat{\in} \omega_\Pi \sqsubseteq K_\Pi, \omega_\Pi \in \eta_{12}$; and V_Π is a $DPS - nbd$ of ς^{α_2}, for $\varsigma^{\alpha_2} \widehat{\in} \hbar_\Pi \sqsubseteq V_\Pi, \hbar_\Pi \in \eta_{12}$.

4 Increasing (Decreasing) Pairwise Soft Closure Operators

In this section, we will discuss the ideas of increasing and decreasing pairwise soft closure and interior operators in a soft bitopological ordered space. We will also examine the basic characteristics of these concepts.

Definition 4.1. *Given a set* $\omega_\Pi \in P(\Upsilon)^\Pi$*, the increasing pairwise soft closure of* ω_Π*, denoted as* $Icl^s_{12}(\omega_\Pi)$*, is the intersection of all increasing pairwise closed soft sets that contain* ω_Π*. i.e.,* $Icl^s_{12}(\omega_\Pi) = \sqcap\{\hbar_\Pi : \hbar_\Pi$ *is* $IPC-$*soft set,* $\omega_\Pi \sqsubseteq \hbar_\Pi\}$*.*

Similarly, the decreasing pairwise soft closure of ω_Π, denoted as $Dcl^s_{12}(\omega_\Pi)$, is the intersection of all decreasing pairwise closed soft sets that contain ω_Π. i.e., $Dcl^s_{12}(\omega_\Pi) = \sqcap\{K_\Pi : K_\Pi \text{ is } DPC-\text{soft set}, \omega_\Pi \sqsubseteq K_\Pi\}$.
Both $Icl^s_{12}(\omega_\Pi)$ and $Dcl^s_{12}(\omega_\Pi)$ are the smallest increasing and decreasing pairwise closed soft sets containing ω_Π respectively.

On the other hand, the increasing pairwise soft interior of ω_Π, denoted as $Iint^s_{12}(\omega_\Pi)$, is the union of all increasing pairwise open soft sets that are contained in ω_Π. i.e.,

$$Iint^s_{12}(\omega_\Pi) = \sqcup\{O_\Pi : O_\Pi \text{ is } IPO-\text{soft set }, O_\Pi \sqsubseteq \omega_\Pi\}.$$

Similarly, the decreasing pairwise soft interior of ω_Π, denoted as $Dint^s_{12}(\omega_\Pi)$, is the union of all decreasing pairwise open soft sets that are contained in ω_Π. i.e., $Dint^s_{12}(\omega_\Pi) = \sqcup\{G_\Pi : G_\Pi \text{ is } DPO-\text{soft set}, G_\Pi \sqsubseteq \omega_\Pi\}$.

Both $Iint^s_{12}(\omega_\Pi)$ and $Dint^s_{12}(\omega_\Pi)$ are the largest increasing and decreasing pairwise open soft sets contained in ω_Π respectively.

Proposition 4.1. *For any soft set ω_Π in an $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, the following holds:*
Case:

1. $[Icl^s_{12}(\omega_\Pi)]^c = Dint^s_{12}(\omega^c_\Pi)$.
2. $[Dcl^s_{12}(\omega_\Pi)]^c = Iint^s_{12}(\omega^c_\Pi)$.
3. $[Dint^s_{12}(\omega_\Pi)]^c = Icl^s_{12}(\omega^c_\Pi)$.
4. $[Iint^s_{12}(\omega_\Pi)]^c = Dcl^s_{12}(\omega^c_\Pi)$.

Proof. We give only proofs of cases (1) and (3) and the cases (2) and (4) can be derived in a similar manner.
Case:

1. $(Icl^s_{12}(\omega_\Pi))^c = (\sqcap\{\lambda_\Pi : \omega_\Pi \sqsubseteq \lambda_\Pi, \lambda_\Pi \text{ is } IPC-\text{soft set}\})^c = \sqcup\{\lambda^c_\Pi : \lambda^c_\Pi \sqsubseteq \omega^c_\Pi, \lambda^c_\Pi \text{ is } IPO-\text{soft set}\} = Dint^s_{12}(\omega^c_\Pi)$. Hence, $[Icl^s_{12}(\omega_\Pi)]^c = Dint^s_{12}(\omega^c_\Pi)$.

3. $(Dint^s_{12}(\omega_\Pi))^c = (\sqcup\{\lambda_\Pi : \lambda_\Pi \sqsubseteq \omega_\Pi, \lambda_\Pi \text{ is } IPO-\text{soft set }\})^c = \sqcap\{\lambda^c_\Pi : \omega^c_\Pi \sqsubseteq \lambda^c_\Pi, \lambda^c_\Pi \text{ is } IPC-\text{soft set}\} = Icl^s_{12}(\omega^c_\Pi)$. Hence, $[Dint^s_{12}(\omega_\Pi)]^c = Icl^s_{12}(\omega^c_\Pi)$.

□

Example 4.1. *In this example, we consider an $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ where $\Pi = \{\alpha_1, \alpha_2\}, \lesssim = \blacktriangle \cup \{(1, 2)\}$ is a partial order relation on the set of real numbers $\Re$ and $\eta_1 = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^1_\Pi \sqsubseteq \Re_\Pi : 1 \in \omega^1_\Pi\}$ and $\eta_2 = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^2_\Pi \sqsubseteq \Re_\Pi : 2 \in \omega^2_\Pi\}$.*

Then, $\eta^I_1 = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^1_\Pi \sqsubseteq \Re_\Pi : 1, 2 \in \omega^1_\Pi\}$, $\eta^D_1 = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^1_\Pi \sqsubseteq \Re_\Pi : 1 \in \omega^1_\Pi\}$, $\eta^I_2 = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^2_\Pi \sqsubseteq \Re_\Pi : 2 \in \omega^2_\Pi\}$ and $\eta^D_2 = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^2_\Pi \sqsubseteq \Re_\Pi : 1, 2 \in \omega^2_\Pi\}$ are increasing and decreasing soft topologies over $\Re$. Clear, $\eta^I_1 \cup \eta^D_1 \subseteq \eta_1$, and $\eta^I_2 \cup \eta^D_2 \subseteq \eta_2$.

Now, $\eta_{12} = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^1_\Pi \sqcup \omega^2_\Pi : 1 \in \omega^1_\Pi, 2 \in \omega^2_\Pi\}$ and $\eta^{IP}_{12} = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^1_\Pi \sqcup \omega^2_\Pi : 1, 2 \in \omega^1_\Pi, 2 \in \omega^2_\Pi\}$,

$\eta^{DP}_{12} = \{\Re_\Pi, \widehat{\phi}\} \cup \{\omega^1_\Pi \sqcup \omega^2_\Pi : 1, 2 \in \omega^1_\Pi, 1 \in \omega^2_\Pi\}$. Clear, $\eta^I_1 \cup \eta^I_2 \subseteq \eta^{IP}_{12}$, $\eta^D_1 \cup \eta^D_2 \subseteq \eta^{DP}_{12}$, and $\eta^{IP}_{12} \cup \eta^{DP}_{12} \subseteq \eta_{12}$.

On the other hand, $\eta_{12}^{cDP} = \eta_{12}^{IPc} = \{\Re_{\Pi}, \widehat{\phi}\} \cup \{\omega_{\Pi}^1 \sqcup \omega_{\Pi}^2 : 1, 2 \notin \omega_{\Pi}^1, 2 \notin \omega_{\Pi}^2\}$ *and* $\eta_{12}^{cIP} = \eta_{12}^{DPc} = \{\Re_{\Pi}, \widehat{\phi}\} \cup \{\omega_{\Pi}^1 \sqcup \omega_{\Pi}^2 : 1, 2 \notin \omega_{\Pi}^1, 1 \notin \omega_{\Pi}^2\}$.

Theorem 4.1. *This theorem states properties of the increasing pairwise soft closure operator* Icl_{12}^s *in an* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$*, specifically in relation to sets* $\omega_{\Pi}, \lambda_{\Pi}$ *in* $P(\Upsilon)^{\Pi}$.
Case:

1. $Icl_{12}^s(\widehat{\phi}) = \widehat{\phi}$ *and* $Icl_{12}^s(\Upsilon_{\Pi}) = \Upsilon_{\Pi}$.
2. $\omega_{\Pi} \sqsubseteq Icl_{12}^s(\omega_{\Pi})$.
3. ω_{Π} *is an* $IPC-$*soft set if and only if* $\omega_{\Pi} = Icl_{12}^s(\omega_{\Pi})$.
4. $\omega_{\Pi} \sqsubseteq \lambda_{\Pi} \Rightarrow Icl_{12}^s(\omega_{\Pi}) \sqsubseteq Icl_{12}^s(\lambda_{\Pi})$.
5. $Icl_{12}^s(\omega_{\Pi}) \sqcup Icl_{12}^s(\lambda_{\Pi}) \sqsubseteq Icl_{12}^s[\omega_{\Pi} \sqcup \lambda_{\Pi}]$.
6. $Icl_{12}^s(Icl_{12}^s(\omega_{\Pi})) = Icl_{12}^s(\omega_{\Pi})$.

Proof. Case:

The proof of (1), (2) and (3) in the theorem are straightforwardly derived from the definition of the closure operator as defined in Definition 4.1.

4. This part of the proof is showing that if soft set ω_{Π} is a soft subset of soft set λ_{Π}, and λ_{Π} is a soft subset of $Icl_{12}^s(\lambda_{\Pi})$, then it follows that ω_{Π} is a soft subset of $Icl_{12}^s(\lambda_{\Pi})$. Since $Icl_{12}^s(\lambda_{\Pi})$ is an $IPC-$soft set and $Icl_{12}^s(\omega_{\Pi})$ is the smallest $IPC-$soft set containing ω_{Π}, it follows that $Icl_{12}^s(\omega_{\Pi})$ must be a soft subset of $Icl_{12}^s(\lambda_{\Pi})$.
5. If ω_{Π} is a soft subset of $\omega_{\Pi} \sqcup \lambda_{\Pi}$ and $\omega_{\Pi} \sqcup \lambda_{\Pi}$ is a soft subset of $Icl_{12}^s[\omega_{\Pi} \sqcup \lambda_{\Pi}]$, then it follows that $Icl_{12}^s(\omega_{\Pi})$ is a soft subset of $Icl_{12}^s[\omega_{\Pi} \sqcup \lambda_{\Pi}]$. Similarly, $Icl_{12}^s(\lambda_{\Pi})$ is also a soft subset of $Icl_{12}^s[\omega_{\Pi} \sqcup \lambda_{\Pi}]$. Therefore, it follows that $Icl_{12}^s(\omega_{\Pi}) \sqcup Icl_{12}^s(\lambda_{\Pi})$ is a soft subset of $Icl_{12}^s[\omega_{\Pi} \sqcup \lambda_{\Pi}]$.
6. Clear.

□

The following theorem can be proven in a similar way using the same method as the previous theorem.

Theorem 4.2. *For any* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$*, and any sets* ω_{Π} *and* λ_{Π} *in* $P(\Upsilon)^{\Pi}$*, the following statements hold:*

1. $Dcl_{12}^s(\widehat{\phi}) = \widehat{\phi}$ *and* $Dcl_{12}^s(\Upsilon_{\Pi}) = \Upsilon_{\Pi}$.
2. $\omega_{\Pi} \sqsubseteq Dcl_{12}^s(\omega_{\Pi})$.
3. ω_{Π} *is a* $DPC-$*soft set if and only if* $Dcl_{12}^s(\omega_{\Pi}) = \omega_{\Pi}$.
4. $\omega_{\Pi} \sqsubseteq \lambda_{\Pi} \Rightarrow Dcl_{12}^s(\omega_{\Pi}) \sqsubseteq Dcl_{12}^s(\lambda_{\Pi})$.
5. $Dcl_{12}^s(\omega_{\Pi}) \sqcup Dcl_{12}^s(\lambda_{\Pi}) \sqsubseteq Dcl_{12}^s[\omega_{\Pi} \sqcup \lambda_{\Pi}]$.
6. $Dcl_{12}^s(Dcl_{12}^s(\omega_{\Pi})) = Dcl_{12}^s(\omega_{\Pi})$.

Theorem 4.3. *For any* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$*, and any soft sets* ω_{Π} *and* λ_{Π} *in* $P(\Upsilon)^{\Pi}$*, the following statements hold:*
Case:

1. $Iint^s_{12}(\widehat{\phi}) = \widehat{\phi}$ *and* $Iint^s_{12}(\Upsilon_\Pi) = \Upsilon_\Pi$.
2. $Iint^s_{12}(\omega_\Pi) \sqsubseteq \omega_\Pi$.
3. ω_Π *is an* $IPO-$*soft set if and only if* $Iint^s_{12}(\omega_\Pi) = \omega_\Pi$.
4. $\omega_\Pi \sqsubseteq \lambda_\Pi \Rightarrow Iint^s_{12}(\omega_\Pi) \sqsubseteq Iint^s_{12}(\lambda_\Pi)$.
5. $Iint^s_{12}[\omega_\Pi \sqcap \lambda_\Pi] \sqsubseteq Iint^s_{12}(\omega_\Pi) \sqcap Iint^s_{12}(\lambda_\Pi)$.
6. $Iint^s_{12}(Iint^s_{12}(\omega_\Pi)) = Iint^s_{12}(\omega_\Pi)$.

Proof. Case:

The proof of the first, second, and third statement in this theorem can be easily derived from Definition 4.1.

4. If a soft set ω_Π is contained in another soft set λ_Π and $Iint^s_{12}(\omega_\Pi)$ is the largest $IPO-$soft set contained within ω_Π, then $Iint^s_{12}(\omega_\Pi)$ is also contained within $Iint^s_{12}(\lambda_\Pi)$ which is the largest $IPO-$soft set contained within λ_Π.

5. For the intersection of soft sets ω_Π and λ_Π, $Iint^s_{12}[\omega_\Pi \sqcap \lambda_\Pi]$ is contained within both $Iint^s_{12}(\omega_\Pi)$ and $Iint^s_{12}(\lambda_\Pi)$, which are the largest $IPO-$soft sets contained within ω_Π and λ_Π, respectively.

6. Obvious.

□

Theorem 4.4. *For any* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, *and any soft sets* ω_Π *and* λ_Π *in* $P(\Upsilon)^\Pi$, *the following statements hold:*

1. $Dint^s_{12}(\widehat{\phi}) = \widehat{\phi}$ *and* $Dint^s_{12}(\Upsilon_\Pi) = \Upsilon_\Pi$.
2. $Dint^s_{12}(\omega_\Pi) \sqsubseteq \omega_\Pi$.
3. ω_Π *is a* $DPO-$*set if and only if* $Dint^s_{12}(\omega_\Pi) = \omega_\Pi$.
4. $\omega_\Pi \sqsubseteq \lambda_\Pi \Rightarrow Dint^s_{12}(\omega_\Pi) \sqsubseteq Dint^s_{12}(\lambda_\Pi)$.
5. $Dint^s_{12}[\omega_\Pi \sqcap \lambda_\Pi] \sqsubseteq Dint^s_{12}(\omega_\Pi) \sqcap Dint^s_{12}(\lambda_\Pi)$.
6. $Dint^s_{12}(Dint^s_{12}(\omega_\Pi)) = Dint^s_{12}(M_\Pi)$.

Proof. It is stated that the proof is similar to that of a previous theorem therefore has not been submitted. □

5 $bi-$Ordered Soft Separation Axioms

The section focuses on the introduction, examination, and investigation of $bi-$ordered soft separation axioms namely $PST_i, PST_i^{\bullet}, PST_i^{*}$, and $PST_i^{**}-$ordered spaces, $(i = 0, 1, 2)$. It explores their properties, provides examples, establishes relationships, and presents results. Additionally, it explores new types of regularity and normality in soft bitopological ordered spaces, highlighting their relationships with other properties, which contributes to a deeper understanding of these spaces.

Definition 5.1. *An $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is said to be:*

1. *Lower pairwise soft T_1-ordered ($LPST_1-$ ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists an $ITPS-$ nbd ε_{Π} of ν such that $\zeta \notin \varepsilon_{\Pi}$.*
2. *Lower pairwise soft $T_1^{\bullet}-$ordered ($LPST_1^{\bullet}-$ ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists an $ITPS-$ nbd ε_{Π} of ν such that $y \not\Subset \varepsilon_{\Pi}$.*
3. *Lower pairwise soft $T_1^{*}-$ordered ($LPST_1^{*}-$ ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists an $IPPS-$ nbd ε_{Π} of ν such that $\zeta \not\Subset \varepsilon_{\Pi}$.*
4. *Lower pairwise soft $T_1^{**}-$ordered ($LPST_1^{**}-$ ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists an $IPPS-$ nbd ε_{Π} of ν such that $\zeta \notin \varepsilon_{\Pi}$.*
5. *Upper pairwise soft T_1-ordered ($UPST_1-$ ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists a $DTPS-$ nbd ε_{Π} of ζ such that $\nu \notin \varepsilon_{\Pi}$.*
6. *Upper pairwise soft $T_1^{\bullet}-$ordered ($UPST_1^{\bullet}-$ ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists a $DTPS-$ nbd ε_{Π} of ζ such that $\nu \not\Subset \varepsilon_{\Pi}$.*
7. *Upper pairwise soft $T_1^{*}-$ordered ($UPST_1^{*}-$ ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists a $DPPS-$ nbd ε_{Π} of ζ such that $\nu \not\Subset \varepsilon_{\Pi}$.*
8. *Upper pairwise soft $T_1^{**}-$ordered ($UPST_1^{**}-$ ordered): For any distinct points ν and ζ in Υ such that $\nu \not\lesssim \zeta$ there exists a $DPPS-$ nbd ε_{Π} of ζ such that $\nu \notin \varepsilon_{\Pi}$.*
9. *PST_0-ordered space: An $SBTOS$ is PST_0-ordered if it satisfies either $LPST_1-$ ordered or $UPST_1-$ ordered.*
10. *$PST_0^{\bullet}-$ordered space: An $SBTOS$ is $PST_0^{\bullet}-$ordered if it satisfies either $LPST_1^{\bullet}-$ ordered or $UPST_1^{\bullet}-$ ordered.*
11. *$PST_0^{*}-$ordered space: An $SBTOS$ is $PST_0^{*}-$ordered if it satisfies either $LPST_1^{*}-$ ordered or $UPST_1^{*}-$ ordered.*
12. *$PST_0^{**}-$ordered space: An $SBTOS$ is $PST_0^{**}-$ordered if it satisfies either $LPST_1^{**}-$ ordered or $UPST_1^{**}-$ ordered.*
13. *PST_1-ordered space if it is $LPST_1-$ ordered and $UPST_1-$ ordered.*
14. *$PST_1^{\bullet}-$ordered space if it is $LPST_1^{\bullet}-$ ordered and $UPST_1^{\bullet}-$ ordered.*
15. *$PST_1^{*}-$ordered space: if it is $LPST_1^{*}-$ ordered and $UPST_1^{*}-$ ordered.*
16. *$PST_1^{**}-$ordered space if it is $LPST_1^{**}-$ ordered and $UPST_1^{**}-$ ordered.*
17. *PST_2-ordered space if for every distinct points ν, ζ in Υ such that $\nu \not\lesssim \zeta$ there exist disjoint $ITPS-$nbd ε_{Π} of ν and $DTPS-$nbd V_{Π} of ζ.*

18. *$PST_2^{\bullet}-$ordered space if for every distinct points ν, ζ in Υ such that $\nu \not\lesssim \zeta$ there exist disjoint $ITPS-$nbd ε_Π of ν and $DPPS-$nbd V_Π of ζ.*
19. *$PST_2^{*}-$ordered space if for every distinct points ν, ζ in Υ such that $\nu \not\lesssim \zeta$ there exist disjoint $IPPS-$nbd ε_Π of ν and $DPPS-$nbd V_Π of ζ.*
20. *$PST_2^{**}-$ordered space if for every distinct points ν, ζ in Υ such that $\nu \not\lesssim \zeta$ there exist disjoint $IPPS-$nbd ε_Π of ν and $DTPS-$nbd V_Π of ζ.*

Proposition 5.1. *Every PST_1(resp. $PST_1^{\bullet}, PST_1^{*}, PST_1^{**}$)$-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is also a PST_0(resp. $PST_0^{\bullet}, PST_0^{*}, PST_0^{**}$)$-$ordered space.*

Proof. The proof is straightforward and follows directly from the Definition 5.1 □

The following example is showing that the converse of the proposition is false by providing a specific counterexample.

Example 5.1. *Let $\Pi = \{e_1, e_2\}, \lesssim = \blacktriangle \cup \{(\nu, \zeta), (\nu, z)\}$ be a partial order relation on $\Upsilon = \{\nu, \zeta, z\}$ and $\eta_1 = \{\widehat{\phi}, \Upsilon_\Pi, \omega_\Pi^1, \omega_\Pi^2, \omega_\Pi^3\}$, $\eta_2 = \{\widehat{\phi}, \Upsilon_\Pi, F_\Pi\}$ where,*

$$\begin{aligned} \omega_\Pi^1 &= \big\{(e_1, \{\zeta\}), (e_2, \{\zeta\})\big\}, \\ \omega_\Pi^2 &= \big\{(e_1, \{z\}), (e_2, \{z\})\big\}, \\ \omega_\Pi^3 &= \big\{(e_1, \{\zeta, z\}), (e_2, \{\zeta, z\})\big\}, \\ F_\Pi &= \big\{(e_1, \{\nu, \zeta\}), (e_2, \{\nu, \zeta\})\big\}. \end{aligned}$$

Then $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $LPST_1$(resp. $LPST_1^{\bullet}, LPST_1^{*}, LPST_1^{**}$)$-$ ordered. So it is PST_0(resp. $PST_0^{\bullet}, PST_0^{*}, PST_0^{**}$)$-$ordered. On the other hand, every decreasing pairwise soft neighborhood of ν containing ζ .

Proposition 5.2. *Every PST_2(resp. PST_2^{**})$-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is also a $PST_1^{\bullet}$(resp. PST_1^{**})$-$ordered space.*

Proof. The proof directly follows from the Definition 5.1. □

The example that is being given is to show that the converse of this proposition is false.

Example 5.2. *By taking $\eta_1 = \eta_2 = \eta$. The example is referring to an Example 4.7 in a previous work, [5]. It is stated that this example is PST_1-ordered (or $PST_1^{**}-$ordered) but not PST_2-ordered (or $PST_2^{**}-$ordered). This means that there exist PST_1-ordered (or $PST_1^{**}-$ordered) spaces that are not PST_2-ordered (or $PST_2^{**}-$ ordered), which contradicts the converse of the proposition.*

Proposition 5.3. *Every $PST_0^{\bullet}$(resp. $PST_1^{\bullet}, PST_0^{*}, PST_1^{*}$)$-$ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is also a PST_0(resp. $PST_1, PST_0^{**}, PST_1^{**}$)$-$ordered space.*

Proof. The proof relies on the observation that if a total non-belong relation $\not\Subset$ exists, then it implies a non-belong relation $\not\in$. □

The provided example serves to illustrate that the converse of this proposition is not true.

Example 5.3. *Let* $\Pi, \lesssim$ *and* Υ *as in Example 5.1 and* $\eta_1 = \{\widehat{\phi}, \Upsilon_\Pi, \omega^1_\Pi, \omega^2_\Pi, \omega^3_\Pi, \omega^4_\Pi\}$, $\eta_2 = \{\widehat{\phi}, \Upsilon_\Pi, F^1_\Pi, F^2_\Pi\}$ *where,*

$$\begin{aligned}
\omega^1_\Pi &= \{(e_1, \{\zeta\}), (e_2, \{\nu, \zeta\})\},\\
\omega^2_\Pi &= \{(e_1, \{z\}), (e_2, \{\nu, z\})\},\\
\omega^3_\Pi &= \{(e_1, \{\zeta, z\}), (e_2, \Upsilon)\},\\
\omega^4_\Pi &= \{(e_1, \emptyset), (e_2, \{\nu\})\},\\
F^1_\Pi &= \{(e_1, \{\nu\}), (e_2, \{\nu, \zeta\})\},\\
F^2_\Pi &= \{(e_1, \emptyset), (e_2, \{\nu, \zeta\})\}.
\end{aligned}$$

Now, $\eta_{12} = \eta_1 \cup \eta_2 \cup \{\lambda^1_\Pi, \lambda^2_\Pi, \lambda^3_\Pi\}$ *where,*

$$\begin{aligned}
\lambda^1_\Pi &= \{(e_1, \{\nu, \zeta\}), (e_2, \{\nu, \zeta\})\},\\
\lambda^2_\Pi &= \{(e_1, \{\nu, z\}), (e_2, \Upsilon)\},\\
\lambda^3_\Pi &= \{(e_1, \{z\}), (e_2, \Upsilon)\}.
\end{aligned}$$

In simple terms, this example is trying to prove that not all $PST^\bullet_0$*(resp.* $PST^\bullet_1, PST^*_0, PST^*_1$*)−ordered spaces are* PST_0*(resp.* $PST_1, PST^{**}_0, PST^{**}_1$*)−ordered spaces, by showing a specific example of a space that is* $PST^\bullet_0$*(resp.* $PST^\bullet_1, PST^*_0, PST^*_1$*)−ordered but not* PST_0*(resp.* $PST_1, PST^{**}_0, PST^{**}_1$*)−ordered.*

Proposition 5.4. *Every* PST_2*−ordered space* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is* PST^*_2*−ordered.*

Proof. The proof for the proposition states that the belong relation $\in$ implies a total belong relation $\Subset$. □

Example 5.4. *Let* $\Pi = \{e_\alpha, e_\beta\}$ *be a set of parameters,* $\lesssim = \blacktriangle \cup \{(1, 2)\}$ *be a partial order relation on the set of natural numbers* $\aleph$*. Define* $\eta_1 = \{\omega_\Pi \sqsubseteq \aleph_\Pi$ *such that* $1 \notin \omega_\Pi\}$ *and* $\eta_2 = \{F_\Pi \sqsubseteq \aleph_\Pi$ *such that* $2 \in \omega_\Pi\}$*. The example states that this specific space is* PST^*_2*−ordered but not* PST_2*−ordered.*

Proposition 5.5. *Every* PST_2*(resp.* PST^{**}_2*)−ordered space* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is* $PST^\bullet_2$*(resp.* PST^*_2*)−ordered.*

Proof. The proof for the proposition states that the belong relation $\in$ implies a total belong relation $\Subset$. □

Example 5.5. *The example provided states that it follows from an earlier example (Example 5.3) that a specific space is* $PST^\bullet_2$*(resp.* PST^*_2*)−ordered but not* PST_2*(resp.* PST^{**}_2*)−ordered.*

Proposition 5.6. *Every* $PST^\bullet_0$*(resp.* $PST^\bullet_1, PST^\bullet_2, PST_2, PST^*_2, PST^{**}_2$*)−ordered space* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is also a* PST^{**}_0*(resp.* $PST^{**}_1, PST^{**}_1, PST^*_1, PST^*_0, PST^{**}_0$*)−ordered space.*

Proof. It is based on the principle that belong relation $\in$ implies a total belong relation $\Subset$ and a total non belong relation $\not\Subset$ implies a non belong relation $\notin$. □

Example 5.6. *It follows from Example 5.3, illustrates that a specific space is* PST^{**}_0*(resp.* $PST^{**}_1, PST^*_1, PST^*_0, PST^{**}_0$*)− ordered but not* $PST^\bullet_0$*(resp.* $PST^\bullet_1, PST^\bullet_2, PST_2, PST^*_2, PST^{**}_2$*)−ordered.*

The diagram illustrates the relationship between different types of separation axioms, as well as the implications between them as described in this paper.

$$
\begin{array}{ccccccc}
PST_1^{\bullet} & \longrightarrow & PST_1 & \longrightarrow & PST_0 & \longrightarrow & PST_0^{\bullet} \\
 & \not\longleftarrow & & \not\longleftarrow & & \not\longleftarrow & \\
\not\uparrow\downarrow & & \uparrow\not\downarrow & & \not\uparrow\not\downarrow & & \not\uparrow\downarrow \\
PST_1^{**} & \not\longrightarrow & PST_1^{*} & \longrightarrow & PST_0^{*} & \longrightarrow & PST_0^{**} \\
 & \longleftarrow & & \not\longleftarrow & & \not\longleftarrow & \\
\uparrow\not\downarrow & & \uparrow\not\downarrow & & \uparrow\not\downarrow & & \uparrow\not\downarrow \\
PST_2^{\bullet} & \not\longrightarrow & PST_2 & \longrightarrow & PST_2^{*} & \not\longrightarrow & PST_2^{**} \\
 & \longleftarrow & & \not\longleftarrow & & \longleftarrow &
\end{array}
$$

Theorem 5.1. *Let* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *be an* $SBTOS$. *Then the following three statements are equivalent:*

1. *The space is* $UPST_1^{\bullet}$*(resp.* $LPST_1^{\bullet}$*)*$-$*ordered,*
2. *For any two elements* ν *and* ζ *in* Υ *such that* $\nu \not\lesssim \zeta$, *there is a* $PO-$*soft set* ω_Π *containing* ζ *(resp.* ν*) in which* $\nu \not\lesssim z$*(resp.* $z \not\lesssim \zeta$*) for every* $z \in \omega_\Pi$,
3. *For any* ν *in* Υ, *the set* $(i(\nu))_\Pi$ *(resp.* $d(\nu)_\Pi$*) is* $PC-$*soft.*

Proof.

$(1 \to 2)$ If $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is an $UPST_1^{\bullet}-$ordered space, and ν and ζ are elements of Υ such that $\nu \not\lesssim \zeta$. Then there exists a $DTPS-$nbd ε_Π of ζ such that $\nu \not\Cap \varepsilon_\Pi$. Putting $\omega_\Pi = sint(\varepsilon_\Pi)$. Suppose that $\omega_\Pi \not\sqsubseteq (i(\nu))^c_\Pi$. Then there exists $z \Cap \omega_\Pi$ and $z \not\Cap (i(\nu))^c_\Pi$. It follows that $z \in (i(\nu))_\Pi$, which implies that $\nu \lesssim z$. Now, $z \Cap \omega_\Pi \sqsubseteq \varepsilon_\Pi$ implies that $\nu \Cap \varepsilon_\Pi$. However, this contradicts the fact that $\nu \not\Cap \varepsilon_\Pi$. Thus $\omega_\Pi \sqsubseteq (i(\nu))^c_\Pi$. Hence $\nu \not\lesssim z$, for every $z \Cap \omega_\Pi$.

$(2 \to 3)$ Consider $\nu \in \Upsilon$ and let $\rho \Cap (i(\nu))^c_\Pi$. Then $\nu \not\lesssim \rho$. Therefore there exists a $PO-$soft set ω_Π containing a such that $\omega_\Pi \sqsubseteq (i(\nu))^c_\Pi$. Given that ν and ρ are picked without any specific criteria, then a pairwise soft set $(i(\nu))^c_\Pi$ is $PO-$ soft, for $\nu \in \Upsilon$. Hence $(i(\nu))_\Pi$ is $PC-$soft, for any $\nu \in \Upsilon$.

$(3 \to 1)$ Let $\nu \not\lesssim \zeta \ \in \Upsilon$. Obviously, $(i(\nu))_\Pi$ is increasing and by hypothesis, $(i(\nu))_\Pi$ is $PC-$ soft. Then $(i(\nu))^c_\Pi$ is a $DPO-$soft soft set satisfies that $\zeta \in (i(\nu))^c_\Pi$ and $\nu \not\Cap (i(\nu))^c_\Pi$. Thus, the proof is finished.

An analogous proof can be applied for the case inside the parentheses. □

Corollary 5.1. *If* ν *is the smallest (resp. the largest) element of a* $LPST_1^{\bullet}$*(resp.* $UPST_1^{\bullet}$*)*$-$*ordered space* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, *then* ν_Π *is* DPC *(resp.* IPC*)*$-$*soft.*

Proposition 5.7. *If* ν *is the smallest (resp. the largest) element of a finite* $PST_1^{\bullet}$ *ordered space* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, *then* ν_Π *is* DPC *(resp.* IPC*)*$-$*soft.*

Proof. The proposition is verified when ν is the smallest element, and the other case can be proved analogously. Since ν is the smallest element of Υ. Then $\nu \lesssim \zeta, \forall \zeta \in \Upsilon$. By the anti-symmetric of $\lesssim$, we have $\zeta \not\lesssim \nu, \forall \zeta \in \Upsilon$. By hypothesis, there is a $DTPS-$nbd F_Π of ν such that $\zeta \notin F_\Pi$. It follows that $\nu_\Pi = \sqcap F_\Pi$. Since Υ is finite, then ν_Π is $DPO-$soft.

A parallel argument can be made for the situation inside the parentheses. □

Proposition 5.8. *If ν is the smallest (resp. the largest) element of a finite a PST_1^*-ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, then F_Π^ν is DPO (resp. IPO)$-$soft.*

Proof. The proof is analogous to Proposition 5.7, with the substitution of ν_Π by F_Π^ν. □

The aforementioned Proposition can be established in the scenario where $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is a finite PST_1^{**}-ordered space.

Proposition 5.9. *A finite $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PST_1^\bullet-$ordered if and only if it is PST_2-ordered.*

Proof.
Necessity: For each $\zeta \in (i(\nu))^c_\Pi$, we have $(d(\zeta))_\Pi$ is $PC-$ soft. Since Υ is finite, then $\sqcup_{\zeta \in (i(\nu))^c_\Pi} d(\zeta)$ is $PC-$ soft. Therefore $(\sqcup_{\zeta \in (i(\nu))^c_\Pi} d(\zeta))^c = (i(\nu))_\Pi$ is a $PO-$ soft set. Thus $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is a PST_2-ordered space.

Sufficiency: It directly follows from Proposition 5.2. □

Proposition 5.10. *Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $SBTOS$ with $\eta_1 = \eta_2 = \eta$. If $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PST_i^\bullet-$ordered, then $(\Upsilon, \eta, \Pi, \lesssim)$ is always $P-$soft T_i-ordered, for $i = 0, 1$.*

Proof. We have shown the proposition when $i = 1$, and the other instance can be shown similarly. Let ν, ζ be two distinct points in $(\Upsilon, \eta, \Pi, \lesssim)$ such that $\nu \lesssim \zeta$. As $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PST_1^\bullet$, then there exist an $ITPS-$nbd ε_Π of ν such that $\zeta \notin \varepsilon_\Pi$ and a $ITPS-$nbd F_Π of ζ such that $\nu \notin F_\Pi$. Since $\eta_1 = \eta_2 = \eta$, then ε_Π is an increasing soft neighborhood of ν such that $\zeta \notin \varepsilon_\Pi$ and F_Π is a decreasing soft neighborhood of ζ such that $\nu \notin F_\Pi$ in $(\Upsilon, \eta, \Pi, \lesssim)$. Thus $(\Upsilon, \eta, \Pi, \lesssim)$ is $P-$soft T_1-ordered. □

Proposition 5.11. *Let $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $SBTOS$ with $\eta_1 = \eta_2 = \eta$. If $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PST_2-ordered, then $(\Upsilon, \eta, \Pi, \lesssim)$ is always $P-$soft T_2-ordered.*

Proof. The proof is analogous to Proposition 5.10. □

Definition 5.2. *Let $\Gamma \subseteq \Upsilon$ and $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ be an $SBTOS$. Then $(\Gamma, \eta_{1\Gamma}, \eta_{2\Gamma}, \Pi, \lesssim_\Gamma)$ is called soft bi$-$ordered subspace of $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ provided that $(\Gamma, \eta_{1\Gamma}, \eta_{2\Gamma}, \Pi)$ is soft bitopological subspace of $(\Upsilon, \eta_1, \eta_2, \Pi)$ and $\lesssim_\Gamma = \lesssim \cap \Gamma \times \Gamma$.*

Lemma 5.1. *If U_Π is an increasing (resp. a decreasing) pairwise soft subset of an $SBTOS (\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, then $U_\Pi \sqcap \Gamma_\Pi$ is an increasing (resp. a decreasing) pairwise soft subset of a soft bi$-$ordered subspace $(\Gamma, \eta_{1\Gamma}, \eta_{2\Gamma}, \Pi, \lesssim_\Gamma)$.*

Proof. Let U_Π be an increasing pairwise soft subset of an $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$.
In a soft $bi-$ordered subspace $(\Gamma, \eta_{1\Gamma}, \eta_{2\Gamma}, \Pi, \lesssim_\Gamma)$, let $\rho \in i_{\lesssim_\Gamma}(U_\Pi \sqcap \Gamma_\Pi)$.
Since $i_{\lesssim_\Gamma}(U_\Pi \sqcap \Gamma_\Pi) \sqsubseteq i_{\lesssim_\Gamma}(U_\Pi) \sqcap i_{\lesssim_\Gamma}(\Gamma_\Pi) \sqsubseteq U_\Pi \sqcap \Gamma_\Pi$, then $\rho \in (U_\Pi \sqcap \Gamma_\Pi)$.

Therefore $i_{\lesssim_\Gamma}(U_\Pi \sqcap \Gamma_\Pi) = U_\Pi \sqcap \Gamma_\Pi$. Thus $U_\Pi \sqcap \Gamma_\Pi$ is an increasing pairwise soft subset of a soft $bi-$ordered subspace $(\Gamma, \eta_{1\Gamma}, \eta_{2\Gamma}, \Pi, \lesssim_\Gamma)$.

The demonstration is parallel in the case where U_Π is decreasing. □

Theorem 5.2. *The property of being a* PST_i *(resp.* $PST_i^\bullet, PST_i^*, PST_i^{**}$*)−ordered space is hereditary, for* $i = 0, 1, 2$.

Proof. We establish the theorem for the case PST_2, and the other cases can be demonstrated in a similar way. Let $(\Gamma, \eta_{1\Gamma}, \eta_{2\Gamma}, \Pi, \lesssim_\Gamma)$ be a soft $bi-$ordered subspace of a PST_2-ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$. If $\rho, \delta \in \Gamma$ such that $\rho \lesssim_\Gamma \delta$, then $\rho \lesssim \delta$. So by hypothesis, there exist disjoint total pairwise soft neighborhoods ε_Π and V_Π of ρ and δ, respectively, such that ε_Π is increasing and V_Π is decreasing. Setting $U_\Pi = \Gamma_\Pi \sqcap \varepsilon_\Pi$ and $\omega_\Pi = \Gamma_\Pi \sqcap V_\Pi$, by Lemma 5.1, we infer that U_Π is an $ITPS-$nbd of ρ and ω_Π is a $DTPS-$nbd of δ. Since the soft neighborhoods U_Π and ω_Π are disjoint, it follows that $(\Gamma, \eta_{1\Gamma}, \eta_{2\Gamma}, \Pi, \lesssim_\Gamma)$ is PST_2-ordered. □

Definition 5.3. *For two soft subsets* ω_Π *and* λ_Π *of an* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$*, we say that* ω_Π *is pairwise soft neighborhood of* λ_Π *provided that there exists a* $PO-$*soft set* F_Π *such that* $\lambda_\Pi \sqsubseteq F_\Pi \sqsubseteq \omega_\Pi$.

Definition 5.4. *An* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is said to be:*

1. *Lower (resp. upper)* $PT-$*soft regularly ordered if for every decreasing (resp. increasing) pairwise closed soft set* λ_Π *and* $\nu \in \Upsilon$ *such that* $\nu \notin \lambda_\Pi$ *there exist disjoint pairwise soft neighborhood* ε_Π *of* λ_Π *and increasing (resp. decreasing) total pairwise soft neighborhood* V_Π *of* ν *such that* ε_Π *is decreasing (resp. increasing).*

2. *Lower (resp. upper)* $PP-$*soft regularly ordered if for every decreasing (resp. increasing) pairwise closed soft set* λ_Π *and* $\nu \in \Upsilon$ *such that* $\nu \notin \lambda_\Pi$ *there exist disjoint pairwise soft neighborhood* ε_Π *of* λ_Π *and increasing (resp. decreasing) partial pairwise soft neighborhood* V_Π *of* ν *such that* ε_Π *is decreasing (resp. increasing).*

3. *Lower (resp. upper)* P^*T-*soft regularly ordered if for every decreasing (resp. increasing) pairwise closed soft set* λ_Π *and* $\nu \in \Upsilon$ *such that* $\nu \notin \lambda_\Pi$ *there exist disjoint pairwise soft neighborhood* ε_Π *of* λ_Π *and increasing (resp. decreasing) total pairwise soft neighborhood* V_Π *of* ν *such that* ε_Π *is decreasing (resp. increasing).*

4. *Lower (resp. upper)* P^*P-*soft regularly ordered if for every decreasing (resp. increasing) pairwise closed soft set* λ_Π *and* $\nu \in \Upsilon$ *such that* $\nu \notin \lambda_\Pi$ *there exist disjoint pairwise soft neighborhood* ε_Π *of* λ_Π *and increasing (resp. decreasing) partial pairwise soft neighborhood* V_Π *of* ν *such that* ε_Π *is decreasing (resp. increasing).*

5. $TP-$*soft regularly ordered if it is both Lower* $PT-$*soft regularly ordered and upper* $PT-$*soft regularly ordered.*

6. $PP-$*soft regularly ordered if it is both Lower* $PP-$*soft regularly ordered and upper* $PP-$*soft regularly ordered.*

7. TP^*-*soft regularly ordered if it is both Lower* P^*T-*soft regularly ordered and upper* P^*T-*soft regularly ordered.*

8. PP^*-*soft regularly ordered if it is both Lower* P^*P-*soft regularly ordered and upper* P^*P-*soft regularly ordered.*

9. *Lower (resp. upper)* $TP-$*soft* T_3 *ordered if it is both* $LPST_1-$*ordered (resp.* $UPST_1-$*ordered) and lower (resp. upper)* $PT-$*soft regularly ordered.*

10. *Lower (resp. upper) $PP-$soft T_3 ordered if it is both $LPST_1^{**}-$ordered (resp. $UPST_1^{**}$-ordered) and lower (resp. upper)) $PP-$soft regularly ordered.*
11. *Lower (resp. upper) $TP^{*}-$soft T_3 ordered if it is both $LPST_1^{\bullet}-$ordered (resp. $UPST_1^{\bullet}-$ordered) and lower (resp. upper) $P^{*}T-$soft regularly ordered.*
12. *Lower (resp. upper) $PP^{*}-$soft T_3 ordered if it is both $LPST_1^{*}-$ordered (resp. $UPST_1^{*}-$ordered) and lower (resp. upper) $P^{*}P-$soft regularly ordered.*
13. *$TP-$soft T_3 ordered if it is both lower $TP-$soft T_3 ordered and upper $TP-$ soft T_3 ordered.*
14. *$PP-$soft T_3 ordered if it is both lower $PP-$soft T_3 ordered and upper $PP-$soft T_3 ordered.*
15. *$TP^{*}-$ soft T_3 ordered if it is both lower $TP^{*}-$soft T_3 ordered and upper $TP^{*}-$soft T_3 ordered.*
16. *$PP^{*}-$soft T_3 ordered if it is both lower $PP^{*}-$soft T_3 ordered and upper $PP^{*}-$soft T_3 ordered.*

Theorem 5.3. *An $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is lower (resp. upper) $PT(P^{*}T)-$soft regularly ordered if and only if for all $\nu \in \Upsilon$ and every increasing (resp. decreasing) pairwise open soft set U_Π containing ν, there is an increasing (resp. decreasing) total pairwise soft neighborhood V_Π of ν satisfies that $cl_{12}^{s}(V_\Pi) \sqsubseteq U_\Pi$.*

Proof.
Necessity: Let $\nu \in \Upsilon$ and U_Π be an $IPO-$soft set partially containing ν. Then, U_Π^c is $DPO-$soft such that $\nu \notin U_\Pi^c$. By hypothesis, there exist disjoint pairwise soft neighborhood ε_Π of U_Π^c and $ITPS-$nbd V_Π of ν. So there is a $PO-$soft set ω_Π such that $U_\Pi^c \sqsubseteq \omega_\Pi \sqsubseteq \varepsilon_\Pi$. Since $V_\Pi \sqsubseteq \varepsilon_\Pi^c$, then $V_\Pi \sqsubseteq \varepsilon_\Pi^c \sqsubseteq \omega_\Pi^c \sqsubseteq U_\Pi$ and since ω_Π^c is $PC-$soft, then $cl_{12}^{s}(V_\Pi) \sqsubseteq \omega_\Pi^c \sqsubseteq U_\Pi$.

Sufficiency: Let $\nu \in \Upsilon$ and λ_Π be a $DPC-$soft set such that $\nu \notin \lambda_\Pi$. Then λ_Π^c be an $IPO-$soft set containing ν. So that, by hypothesis, there is an $ITPS-$nbd V_Π of ν such that $cl_{12}^{s}(V_\Pi) \sqsubseteq \lambda_\Pi^c$. Consequently, $(cl_{12}^{s}(V_\Pi))^c$ is a $PO-$soft set containing λ_Π. Thus $d((cl_{12}^{s}(V_\Pi))^c)$ is a pairwise soft neighborhood and decreasing of λ_Π. Suppose that $V_\Pi \sqcap d((cl_{12}^{s}(V_\Pi))^c) \neq \widehat{\phi}$. Then there exists $z \in \Upsilon$ such that $z \in V_\Pi$ and $z \in d((cl_{12}^{s}(V_\Pi))^c)$. So there exists $\zeta \in ((cl_{12}^{s}(V_\Pi))^c(\alpha)$ satisfies that $z \lesssim \zeta$. This means that $\zeta \in V(\alpha)$. But this contradicts the disjointedness between V_Π and $(cl_{12}^{s}(V_\Pi))^c$. Thus $V_\Pi \sqcap d((cl_{12}^{s}(V_\Pi))^c) = \widehat{\phi}$. This completes the proof.

A similar proof can be given for the case between parentheses. □

Theorem 5.4. *An $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is lower (resp. upper) PP $(P^{*}P)-$soft regularly ordered if and only if for all $\nu \in \Upsilon$ and every increasing (resp. decreasing) pairwise open soft set U_Π containing ν, there is an increasing (resp. decreasing) partial pairwise soft neighborhood V_Π of ν satisfies that $cl_{12}^{s}(V_\Pi) \sqsubseteq U_\Pi$.*

Proof. The proof is similar to the proof of Theorem 5.3. □

Proposition 5.12. *Every $TP-$ soft T_3-ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PP-$soft T_3-ordered.*

Proof. The proposition's proof establishes that the belong relation, denoted by $\in$, can be extended to a partial belong relation denoted by $\Subset$. □

Example 5.7. *Let $\Pi = \{e_\alpha, e_\beta, e_\gamma\}$ be a set of parameters, $\lesssim = \blacktriangle \cup \{(1,2)\}$ be a partial order relation on the set of natural numbers $\aleph$. Define, $\eta_1 = \{\omega_\Pi \sqsubseteq \aleph_\Pi$ such that $1 \not\Subset \omega_\Pi$ or $[1 \in \omega(e_\beta)$ and ω_Π^c is finite $]\}$ and $\eta_2 = \{F_\Pi \sqsubseteq \aleph_\Pi$ such that $3 \in F(e_\alpha)$, and F_Π^c is finite$\}$. Obviously, $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ is $PP-$ soft T_3-ordered . A soft subset λ_Π of $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ is a decreasing pairwise closed soft set if $[1 \in \lambda_\Pi$ and λ_Π is infinite $]$ or $[1 \notin \lambda(e_\beta),\ 3 \notin \lambda(e_\alpha)$ and λ_Π is finite $]$. To illustrate that $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ is not lower $PT-$soft regularly ordered, we define a decreasing soft closed set λ_Π as follows:*
$\lambda_\Pi = \{(e_\alpha, \{1,2\}), (e_\beta, \{3\}), (e_\gamma, \{1,2\})\}$.

Since $1 \not\Subset \lambda_\Pi$ and there do not exist disjoint soft neighborhoods ε_Π and V_Π containing λ_Π and 1, respectively, then $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ is not lower $PT-$soft regularly ordered. Hence $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ is not $TP-$soft T_3-ordered.

Proposition 5.13. *An $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $TP-$soft T_3-ordered if and only if TP^*-soft T_3-ordered.*

Proof. On the one hand, $\nu \not\Subset \omega_\Pi$ implies that $\nu \notin \omega_\Pi$, then $TP-$soft T_3-ordered implies TP^*-soft T_3-ordered. On the other hand, the definition of TP^*-soft T_3-ordered implies that for every decreasing (resp. increasing) pairwise closed soft set λ_Π and $\nu \in \Upsilon$ such that $\nu \notin \lambda_\Pi$, there exist disjoint pairwise soft neighborhood ε_Π of λ_Π and increasing (resp. decreasing) total pairwise soft neighborhood V_Π of ν, such that ε_Π is decreasing (resp. increasing). Since ε_Π and V_Π are disjoint, then $\nu \not\Subset \lambda_\Pi$ and $\forall \nu \not\lesssim \zeta$, there exist an $ITPS$ $\hbar_\Pi$ of ν such that $\zeta \not\Subset \hbar_\Pi$. Hence the definitions of $TP-$soft T_3-ordered and TP^*-soft T_3-ordered are equivalent. □

Corollary 5.2. *An $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PP-$soft T_3-ordered if and only if PP^*-soft T_3-ordered.*

Proposition 5.14. *Every TP^*- soft T_3-ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PP^*-soft T_3-ordered.*

Proof. The proof for the proposition states that the belong relation $\in$ implies a partial belong relation $\Subset$. □

Example 5.8. *From Example 5.7, an $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PP^*-soft T_3-ordered but it is not TP^*-soft T_3-ordered.*

Proposition 5.15. *The following three properties are equivalent if $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is TP^*-soft regularly ordered:*

1. *$(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PST_2-ordered;*
2. *$(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PST_1-ordered;*
3. *$(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PST_0-ordered.*

Proof. The direction $1) \to 2) \to 3)$ is obvious from Propositions 5.1, 5.2, 5.3.

To prove $3) \to 1)$, let $\nu, \zeta \in \Upsilon$ such that $\nu \not\lesssim \zeta$. Since $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PST_0-ordered, then it is lower pairwise soft T_1-ordered or upper pairwise soft T_1-ordered. Say it is upper pairwise soft T_1-ordered. From Theorem 5.1, we have that $(i(\nu))_\Pi$ is $PC-$soft. Obviously, $(i(\nu))_\Pi$ is increasing and $\zeta \not\Subset (i(\nu))_\Pi$. Since $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is TP^*-soft regularly ordered, then there exist disjoint $DTPS-$nbd ε_Π of ζ and pairwise soft neighborhood and increasing V_Π of $(i(\nu))_\Pi$ so V_Π is $ITPS-$nbd of ν. Thus $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is PST_2-ordered. □

Corollary 5.3. *The following three properties are equivalent if $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is lower (upper) P^*T-soft regularly ordered:*

1. $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is* PST_2-*ordered;*
2. $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is* PST_1-*ordered;*
3. $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is* $LPST_1($ *resp.* $UPST_1)-$*ordered.*

Proposition 5.16. *Every* TP^*-*soft* T_3-*ordered space* $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is also* PST_2-*ordered.*

Proof. Proposition 5.15 implies that any TP^*-soft T_3-ordered space is also PST_2-ordered. □

Here is an illustration that shows that the converse of Proposition 5.16 is not necessarily true.

Example 5.9. *Let* $\Pi = \{e_\alpha, e_\beta\}$ *be a set of parameters,* $\lesssim = \blacktriangle \cup \{(1,2)\}$ *be a partial order relation on the set of natural numbers* $\aleph$*. Define* $\eta_1 = \{\omega_\Pi \sqsubseteq \aleph_\Pi$ *such that* $1 \in \omega_\Pi$ *and* ω_Π^c *is infinite* $\}$ *and* $\eta_2 = \{F_\Pi \sqsubseteq \aleph_\Pi$ *such that* $1 \in F_\Pi^c\} \cup \aleph_\Pi$*. Then* $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ *is a soft bitopological ordered space. Obviously,* $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ *is* PST_2-*ordered. We have the following 6 cases:*
For $\nu, \zeta \in \aleph - \{1,2\},\ \nu \neq \zeta:$

1. *Either* $1 \not\lesssim \nu$*. Then we define two soft sets* ε_Π *and* V_Π *as follows* $\varepsilon_\Pi = \{(e_\alpha, \{1,2\}), (e_\beta, \{1,2\})\}$ *and* $V_\Pi = \{(e_\alpha, \{\nu\}), (e_\beta, \{\nu\})\}$*. So* ε_Π *is an* $ITPS-$*nbd of* 1*,* V_Π *is a* $DTPS-$*nbd of* ν *and* $\varepsilon_\Pi \sqcap V_\Pi = \widehat{\phi}$*.*
2. *Or* $\nu \not\lesssim 1$*. Then we define two soft sets* ε_Π *and* V_Π *as follows* $\varepsilon_\Pi = \{(e_\alpha, \{\nu\}), (e_\beta, \{\nu\})\}$ *and* $V_\Pi = \{(e_\alpha, \{1\}), (e_\beta, \{1\})\}$*. So* ε_Π *is an* $ITPS-$*nbd of* ν*,* V_Π *is a* $DTPS-$*nbd of* 1 *and* $\varepsilon_\Pi \sqcap V_\Pi = \widehat{\phi}$*.*
3. *Or* $2 \not\lesssim \nu$*. Then we define two soft sets* ε_Π *and* V_Π *as follows* $\varepsilon_\Pi = \{(e_\alpha, \{2\}), (e_\beta, \{2\})\}$ *and* $V_\Pi = \{(e_\alpha, \{\nu\}), (e_\beta, \{\nu\})\}$*. So* ε_Π *is an* $ITPS-$*nbd of* 2*,* V_Π *is a* $DTPS-$*nbd of* ν *and* $\varepsilon_\Pi \sqcap V_\Pi = \widehat{\phi}$*.*
4. *Or* $2 \not\lesssim 1$*. Then we define two soft sets* ε_Π *and* V_Π *as follows* $\varepsilon_\Pi = \{(e_\alpha, \{2\}), (e_\beta, \{2\})\}$ *and* $V_\Pi = \{(e_\alpha, \{1\}), (e_\beta, \{1\})\}$*. So* ε_Π *is an* $ITPS-$*nbd of* 2*,* V_Π *is a* $DTPS-$*nbd of* 1 *and* $\varepsilon_\Pi \sqcap V_\Pi = \widehat{\phi}$*.*
5. *Or* $\nu \not\lesssim 2$*. Then we define two soft sets* ε_Π *and* V_Π *as follows* $\varepsilon_\Pi = \{(e_\alpha, \{\nu\}), (e_\beta, \{\nu\})\}$ *and* $V_\Pi = \{(e_\alpha, \{1,2\}), (e_\beta, \{1,2\})\}$*. So* ε_Π *is an* $ITPS-$*nbd of* ν*,* V_Π *is a* $DTPS-$*nbd of* 2 *and* $\varepsilon_\Pi \sqcap V_\Pi = \widehat{\phi}$*.*
6. *Or* $\nu \not\lesssim \zeta$*. Then we define two soft sets* ε_Π *and* V_Π *as follows* $\varepsilon_\Pi = \{(e_\alpha, \{\nu\}), (e_\beta, \{\nu\})\}$ *and* $V_\Pi = \{(e_\alpha, \{\zeta\}), (e_\beta, \{\zeta\})\}$*. So* ε_Π *is an* $ITPS-$*nbd of* ν*,* V_Π *is a* $DTPS-$*nbd of* ζ *and* $\varepsilon_\Pi \sqcap V_\Pi = \widehat{\phi}$*.*

To illustrate that $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ *is not lower* P^*T-*soft regularly ordered, we define a* $DPC-$*soft set* λ_Π *as follows:* $\lambda_\Pi = \{(e_\alpha, \{1,2,4,5,...\}), (e_\beta, \{1,2,4,5,...\})\}$*. Since* $3 \notin \lambda_\Pi$ *and there do not exit disjoint pairwise soft neighborhoods* ε_Π *and* V_Π *of* λ_Π *and* 3*, respectively, then* $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ *is not lower* P^*T-*soft regularly ordered, which implies that* $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ *is not* TP^*-*soft* T_3-*ordered.*

Definition 5.5. *An* $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ *is said to be:*

1. *Soft pairwise normally ordered if for each disjoint* $PC-$*soft sets* F_Π *and* λ_Π *such that* F_Π *is increasing and* λ_Π *is decreasing, there exist disjoint pairwise soft neighborhoods* ε_Π *of* F_Π *and* V_Π *of* λ_Π *such that* ε_Π *is increasing and* V_Π *is decreasing.*

2. $TP-$soft T_4-ordered if it is soft pairwise normally ordered and $PST_1^{\bullet}-$ ordered.

Theorem 5.5. *An $SBTOS$ $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is soft pairwise normally ordered if and only if for every decreasing (increasing) pairwise closed soft set F_Π and every decreasing (increasing) pairwise soft neighborhood U_Π of F_Π, there is a decreasing (increasing) pairwise soft neighborhood V_Π of F_Π, satisfies that $cl^s_{12}(V_\Pi) \sqsubseteq U_\Pi$.*

Proof.
Necessity: Let F_Π be a $DPC-$soft set and U_Π be a pairwise soft neighborhood and decreasing of F_Π. Then, U^c_Π is an $IPC-$soft set and $F_\Pi \sqcap U^c_\Pi = \widehat{\phi}$. Since $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is soft pairwise normally ordered, then there exist disjoint pairwise soft neighborhood V_Π of F_Π and pairwise soft neighborhood ε_Π of U^c_Π. Since ε_Π is a pairwise soft neighborhood of U^c_Π, then there exists a $PC-$soft set λ_Π such that $U^c_\Pi \sqsubseteq \lambda_\Pi \sqsubseteq \varepsilon_\Pi$. Consequently, $\varepsilon^c_\Pi \sqsubseteq \lambda^c_\Pi \sqsubseteq U_\Pi$ and $V_\Pi \sqsubseteq \varepsilon^c_\Pi$. So it follow that $cl^s_{12}(V_\Pi) \sqsubseteq cl^s_{12}(\varepsilon^c_\Pi) \sqsubseteq \lambda^c_\Pi \sqsubseteq U_\Pi$. Thus $F_\Pi \sqsubseteq cl^s_{12}(V_\Pi) \sqsubseteq cl^s_{12}(\varepsilon^c_\Pi) \sqsubseteq \lambda^c_\Pi \sqsubseteq U_\Pi$. Hence the necessity part holds.

Sufficiency: Let F^1_Π and F^2_Π be two disjoint $PC-$soft sets such that F^1_Π is decreasing and F^2_Π is increasing. Then F^{2c}_Π is a $DPO-$soft set containing F^1_Π. By hypothesis, there exists a decreasing pairwise soft neighborhood V_Π of F^1_Π such that $cl^s_{12}(V_\Pi) \sqsubseteq F^{2c}_\Pi$. Setting $\lambda_\Pi = \Upsilon_\Pi - cl^s_{12}(V_\Pi)$. This means that λ_Π is a $PO-$soft set containing F^2_Π. Obviously, $F^2_\Pi \sqsubseteq \lambda_\Pi,\ F^1_\Pi \sqsubseteq V_\Pi$ and $\lambda_\Pi \sqcap V_\Pi = \widehat{\phi}$. Now, $i(\lambda_\Pi)$ is a pairwise soft neighborhood and increasing of F^2_Π. Suppose that $i(\lambda_\Pi) \sqcap V_\Pi \neq \widehat{\phi}$. Then there exists $\alpha \in \Pi$ and $\nu \in \Upsilon$ such that $\nu \in i(\lambda_\Pi)$ and $\nu \in V(\alpha) = d(V(\alpha))$. This implies that there exist $\rho \in \lambda(\alpha)$ and $\delta \in V(\alpha)$ such that $\rho \lesssim \nu$ and $\nu \lesssim \delta$. As $\lesssim$ is transitive, then $\rho \lesssim \delta$. Therefore $\delta \Subset \lambda_\Pi \sqcap V_\Pi$. This contradicts the disjointness between λ_Π and V_Π. Thus $i(\lambda_\Pi) \sqcap V_\Pi = \widehat{\phi}$. Hence the proof is completed.

A proof similar can be given for the statement inside the parentheses. □

Proposition 5.17. *Every $TP-$soft T_4-ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is also TP^*-soft T_3-ordered.*

Proof. Let $\rho \in \Upsilon$ and F_Π be a $DPC-$soft set such that $\rho \Subset F_\Pi$. Since $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is $PST^{\bullet}-$ordered, then $(i(\rho))_\Pi$ is an $IPC-$soft set and since $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is soft pairwise normally ordered, then there exist disjoint pairwise soft neighborhood ε_Π and V_Π of $(i(\rho))_\Pi$ and F_Π respectively, such that ε_Π is increasing and V_Π is decreasing. Therefore, $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is lower P^*T-soft regularly ordered. If F_Π is an $IPC-$soft set, then we prove similarly that $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is upper P^*T-soft regularly ordered. Thus $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is TP^*- soft regularly ordered. Hence $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$ is TP^*- soft T_3-ordered. □

The converse of the above proposition is not always true as illustrated in the following example.

Example 5.10. *From Example (4.28) in [5], if we take $\eta_1 = \{\omega_\Pi \sqsubseteq \aleph_\Pi$ such that $1 \Subset \omega_\Pi\}$ and $\eta_2 = \{F_\Pi \sqsubseteq \aleph_\Pi$ such that $1 \in F(\alpha_2)$ and F^c_Π is finite $\}$. Then we have $(\aleph, \eta_1, \eta_2, \Pi, \lesssim)$ is TP^*-soft T_3-ordered, but it is not $TP-$ soft T_4-ordered.*

Theorem 5.6. *The property of being a PST_i $(PST^{\bullet}_i, PST^*_i, PST^{**}_i)-$ordered space is soft bitopological ordered property, for $i = 0, 1, 2$.*

Proof. We prove the theorem in case of PST_2 and the other follow similar lines.

Suppose that ϕ_ψ is an ordered embedding soft homeomorphism map of a PST_2-ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim_1)$ on to an $SBTOS$ $(\Gamma, \eta^*_1, \eta^*_2, K, \lesssim_2)$ and let $\nu, \zeta \in \Gamma$ such that $\nu \not\lesssim_2 \zeta$. Then,

$\nu^\alpha \not\lesssim_2 \zeta^\alpha$, $\forall \alpha \in K$. Since ϕ_ψ is bijective, then there exist ρ^β and δ^β in Υ_Π such that $\phi_\psi(\rho^\beta) = \nu^\alpha$ and $\phi_\psi(\delta^\beta) = \zeta^\alpha$ and since ϕ_ψ is ordered embedding, then $\rho^\beta \not\lesssim_1 \delta^\beta$. So $\rho \not\lesssim_1 b$. By hypothesis, there exist disjoint pairwise soft neighborhoods ε_Π and V_Π of ρ and δ, respectively, such that ε_Π is increasing and V_Π is decreasing. Since ϕ_ψ is bijective soft open, then $\phi_\psi(\varepsilon_\Pi)$ and $\phi_\psi(V_\Pi)$ are disjoint soft neighborhoods of ν and ζ, respectively. It follows by Theorem 2.3, that $\phi_\psi(\varepsilon_\Pi)$ is increasing and $\phi_\psi(V_\Pi)$ is decreasing. This completes the proof. □

Theorem 5.7. *The property of being a $TP^*(PP^*)-$soft T_3-ordered space is soft bitopological ordered property.*

Proof. The proof is similar to the previous theorem □

Theorem 5.8. *The property of being a $TP-$soft T_4-ordered space is soft bitopological ordered property.*

Proof. Suppose that ϕ_ψ is an ordered embedding soft homeomorphism map of a soft pairwise normally ordered space $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim_1)$ on to an $SBTOS$ $(\Gamma, \eta_1^*, \eta_2^*, K, \lesssim_2)$ and let λ_Π and F_Π be two disjoint pairwise closed soft sets such that λ_Π is increasing and F_Π is decreasing. Since ϕ_ψ is bijective soft continuous, then $\phi_\psi^{-1}(\lambda_\Pi)$ and $\phi_\psi^{-1}(F_\Pi)$ are disjoint $PC-$soft sets and since ϕ_ψ is ordered embedding, then $\phi_\psi^{-1}(\lambda_\Pi)$ is increasing and $\phi_\psi^{-1}(F_\Pi)$ is decreasing. By hypothesis, there exist disjoint pairwise soft neighborhoods ε_Π and V_Π of $\phi_\psi^{-1}(\lambda_\Pi)$ and $\phi_\psi^{-1}(F_\Pi)$, respectively, such that ε_Π is increasing and V_Π is decreasing. So $\lambda_\Pi \sqsubseteq \phi_\psi(\varepsilon_\Pi)$ and $F_\Pi \sqsubseteq \phi_\psi(V_\Pi)$. The disjointness of the soft neighborhoods $\phi_\psi(\varepsilon_\Pi)$ and $\phi_\psi(V_\Pi)$ completes the proof. □

6 Discussion

This paper presents the notion of decreasing and increasing pairwise soft sets and investigates various associated properties. Notably, it is shown that the relative complement of an increasing or decreasing pairwise soft set preserves the respective property. The main contribution of this work is the construction of a Soft Bitopological Ordered Space ($SBTOS$); $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$, which refines the given Soft Bitopological Space ($SBTS$); $(\Upsilon, \eta_1, \eta_2, \Pi)$ by introducing a partial order relation on the universe set Υ. New ordered soft separation axioms, namely PST_i-ordered spaces, $PST_i^\bullet$-ordered spaces, PST_i^*-ordered spaces, and PST_i^{**}-ordered spaces, where $i = 0, 1, 2$, are introduced and shown to be strictly stronger than P-soft T_i-ordered spaces as established by El-Shafei et al. in 2019. In Theorem 3.2, it is demonstrated that the collection of increasing or decreasing open soft sets forms an increasing or decreasing soft topology, respectively. Additionally, Proposition 5.15 investigates the conditions under which these PST_i-ordered spaces, with $i = 0, 1, 2$, are equivalent.

Furthermore, the concept of a $bi-$ordered subspace is introduced and its hereditary property within the framework of soft bitopological ordered spaces is examined. Soft bitopological ordered properties are defined and their validity is confirmed for PST_i-ordered spaces, $PST_i^\bullet$-ordered spaces, PST_i^*-ordered spaces, and PST_i^{**}-ordered spaces, where $i = 0, 1, 2$. Moreover, the property of being a TP-soft T_3-ordered space is established as a soft bitopological ordered property. The findings of this study have implications for the interpretation of an $SBTOS$; $(\Upsilon, \eta_1, \eta_2, \Pi, \lesssim)$. It can be regarded as a Soft Topological Space (STS) when $\lesssim$ is an equality relation and $\eta_1 = \eta_2$. Similarly, it can be considered a topological ordered space if Π is a singleton set and $\eta_1 = \eta_2$. Furthermore, an $SBTOS$ exhibits characteristics of a soft bitopological space when Π is a singleton set and $\lesssim$ is an equality relation.

Overall, the concepts introduced and the results obtained in this paper lay the groundwork for further significant research in the field of soft bitopological ordered spaces. Future research directions will include the exploration of pairwise continuity in such spaces. By discussing the obtained results and their interpretations, as well as their implications in the broader context of previous studies and working hypotheses, this paper contributes to the advancement of knowledge in this area.

7 Conclusion

In 1965, Nachbin [24] introduced the concept of topological ordered space, which combines the properties of partial order relations and topological spaces. Later, in 1999, Molodtsov [23] proposed the idea of "soft sets" to address issues related to uncertainty, vagueness, imprecision, and incomplete data. Building upon these concepts, Ittanagi [16] introduced the notion of a soft bitopological space.

In this paper, we introduced the concept of soft bitopological ordered spaces and established some properties of them. We also introduced and studied the notions of increasing (decreasing, balancing) pairwise open (closed) soft sets, increasing (decreasing, balancing) total (partial) pairwise soft neighborhoods, and increasing (decreasing, balancing) pairwise open soft neighborhoods. Additionally, we discussed the origins of increasing (decreasing) pairwise soft closure (interior). This research is an important step towards understanding the properties of soft bitopological ordered spaces and their potential applications in decision making. Future work will focus on exploring these applications in more depth. Through this research, a new class of $bi-$ordered soft separation axioms, called PST_i, $PST_i^{\bullet}$, PST_i^{*}, and PST_i^{**} has been introduced and studied for $(i = 0, 1, 2)$. The concepts of belong, non-belong, partial belong, and total non-belong have been considered to understand their relationships. To aid in understanding, examples have been provided. In future research, we aim to explore new $bi-$ordered soft separation axioms by utilizing these concepts on supra soft topological spaces. We hope that this work will inspire further research and advancements in the field of soft topology.

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